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The Residue of Vector Sets with Applications to Decidability Problems in Petri Nets

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Summary. A set K of integer vectors is called right-closed, if for any element $\underline{m} \in K$ all vectors $\underline{m}' \geq \underline{m}$ are also contained in K. In such a case K is a semilinear set of vectors having a minimal generating set res(K), called the residue of K. A general method is given for computing the residue set of a right-closed set, provided it satisfies a certain decidability criterion.

Various right-closed sets wich are important for analyzing, constructing, or controlling Petri nets are studied. One such set is the set CONTINUAL(T) of all such markings which have an infinite continuation using each transition infinitely many times. It is shown that the residue set of CONTINUAL(T) can be constructed effectively, solving an open problem of Schroff. The proof also solves problem 24 (iii) in the EATCS-Bulletin. The new methods developed in this paper can also be used to show that it is decidable, whether a signal net is prompt [23] and whether certain ω-languages of a Petri net are empty or not.

It is shown, how the behaviour of a given Petri net can be controlled in a simple way in order to realize its maximal central subbehaviour, thereby solving a problem of Nivat and Arnold, or its maximal live subbehaviour as well. This latter approach is used to give a new solution for the bankers problem described by Dijkstra.

Since the restriction imposed on a Petri net by a fact [11] can be formulated as a right closed set, our method also gives a new general approach for „implementations" of facts.

1. Introduction

The basis of many decision procedures in vector addition systems or Petri nets is the so called "property of monotonicity". To give an example: if a sequence of transitions can fire in a given marking, this must also be possible in any marking that is (componentwise) not smaller. In particular, a marking is unbounded if for any integer n there is a place p and a firing sequence w, such that firing w in $\underline{m}$ brings more than n tokens to p. Consequently, unboundedness is a monotone property of markings.

A marking is called dead if any firing inevitably results in a total deadlock. Hence the property of a marking to be not dead is also monotone. This property can be rephrased as follows: $\underline{m}$ is not dead, if an infinite sequence of transitions can fire in $\underline{m}$. Being interested in some particular set $\hat{T} \subseteq T$ of transitions to be fired infinitely often we define: a marking $\underline{m}$ is $\hat{T}$-continual, if an infinite sequence of transitions can fire in $\underline{m}$ containing each $t \in \hat{T}$ infinitely often. $\hat{T}$-continuality is again a monotone property of markings.

To have control on the behaviour of a concurrent system, given by a Petri net, one may wish to know all markings having an undesired property, (e.g. to be unbounded, to be dead). The main purpose of this paper is to show, how finite representations of monotone marking sets can be effectively computed.

To give finite representations of infinite sets of integer vectors we will use the notions of regular and semilinear sets. It was proved in [6] and [10] that these two notions are equivalent.

According to [12], a subset $K \subseteq \mathbb{N}^k$ is called right-closed, if with $\underline{m} \in K$ each $\underline{m}' \geq \underline{m}$ is also contained in K. It is wellknown that the set of minimal elements of such a set is finite and is here called the residue res(K) of K. If $K \subseteq \mathbb{N}^k$ is right-closed and satisfies a particular decidable property, called RES, then res(K) can be effectively computed. In section 2 we give the algorithm and prove its correctness. The results of this section are very general and not specific for Petri nets or vector addition systems.

In section 3 we define place transition nets (P/T-nets) and the notions of bounded, dead, $\hat{T}$-blocked, and $\hat{T}$-continual markings.

The sets UNBOUNDED (NOTDEAD, NOTBLOCKED($\hat{T}$), CONTINUAL($\hat{T}$), resp.) of unbounded (not dead, not $\hat{T}$-blocked, $\hat{T}$-continual, resp.) markings are right-closed sets which satisfy property RES. Hence we can apply the results of section 2 to effectively compute the residue of these sets for a given P/T-net.

In section 4 we use residue sets res(K) to control the behaviour of a P/T-net N in such a way that all reachable markings are in K. The 'control' is completely integrated in the P/T-net and yields a new P/T-net N_K with the same number of places, but possibly additional transitions.

The construction of N_K is also a new method for the "implementation" of facts in P/T-nets in the sense of [11].

In section 5 we then apply the construction to the right-closed sets K of not-dead and T-continual markings. Using the notion of transition systems we show that N_K has the maximal subbehaviour with respect to well defined properties. Of particular interest is the net N_K where K is the set of $\hat{T}$-continual markings. N_K allows exactly the "live" firings of N and prevents from "non live" situations. These results give a solution to a problem of [22] to realize the maximal "central" subbehaviour of processes. We also show how Dijkstra's wellknown banker's problem obtains a new solution.

In section 6 we show that the old problem of a transition to be "hot" is decidable and give applications of this result to the problem of promptness in P/T-nets. Also the emptyness problem for classes $\mathscr{B}^i_\omega$ and $\mathscr{K}^i_\omega$ of ω-behaviour of P/T-nets as introduced in [28, 5] is shown to be decidable.

We acknowledge the work of Schroff [25], who first gave an algorithm to compute the residue res(NOTDEAD). His algorithm was not published and is – compared with ours – very complicated. A result very similar to Theorem 2.13 is contained in [12, Lemma 3], however, the algorithm to compute res(K) for a right-closed set K given there is not very practical since it works by mere enumeration of the two sets K and $\mathbb{N}^k$-K. In [26] our result for effectively computing the set res(CONTINUAL(T)) is mentioned as an open problem. This result and applications to the set of unbounded markings, promptness and the maximal live subbehaviour of a given P/T-net were first derived in [27], but with again unnecessarily complicated proofs.

2. Finite Representation of Integer Vector Sets

Definition 2.1. Let $\mathbb{Z}$ denote the set of integers and $\mathbb{N}$ be the set of nonnegative integers. If A, $B \in \mathbb{Z}^k$ then we write $A+B := \{\underline{x}+\underline{y} \mid \underline{x} \in A, \underline{y} \in B\}$ and

$$A^{\circledast} := \bigcup_{i=0} A_i, \quad \text{where} \quad A_0 := \{\underline{0}\} \quad \text{and} \quad A_{i+1} := A_i + A,$$

and $\underline{0} := (0, \dots, 0)$ is the zero vector of appropriate dimension. Usually vectors are written as small, underlined letters and are understood as column-vectors even though we sometimes prefer to write them as row-vectors, especially in examples.

Definition 2.2. The regular subsets of $\mathbb{Z}^k$ are defined as follows:

(a) Every finite subset of $\mathbb{Z}^k$ is regular.

(b) If A and B are regular subsets of $\mathbb{Z}^k$ so are $A \cup B$, $A+B$, and $A^{\circledast}$.

(c) A set $A \subseteq \mathbb{Z}^k$ is regular, iff it is so by finitely many applications of rules (a) and (b).

Definition 2.3. A regular set $R \subseteq \mathbb{Z}^k$ is called *linear*, if it is of the form $R = \{\underline{x}\} + B^{\circledast}$ for some $\underline{x} \in \mathbb{Z}^k$ and some finite subset $B \subseteq \mathbb{Z}^k$.

A regular set $R \subseteq \mathbb{Z}^k$ is called *semilinear*, if it is a finite union of linear sets.

Theorem 2.4 [6, 10]. *The regular subsets of $\mathbb{Z}^k$ (or $\mathbb{N}^k$) are precisely the semilinear subsets of $\mathbb{Z}^k$ (or $\mathbb{N}^k$).*

Definition 2.5. Let $\mathbb{N}_\omega := \{\omega\} \cup \mathbb{N}$, where ω is a new element satisfying:

$$\forall n \in \mathbb{N}: n < \omega, \ \forall n \in \mathbb{N}_\omega: n + \omega := \omega - n := \omega, \ \min(n, \omega) := n,$$
$$\max(n, \omega) := \omega, \ (n+1) \cdot \omega := \omega, \ 0 \cdot \omega := \omega \cdot 0 := 0.$$

The relations $\geqq$, $\leqq$, $=$ for vectors are understood componentwise and $\underline{x} \lneqq \underline{y}$ is a shorthand for ($x \leqq y$ and $x \neq y$). The dyadic operations $+$, $-$, min, and max are evaluated componentwise too.

For sets M, $M' \subseteq \mathbb{N}_\omega^k$ define:

$$\max(M, M') := \{\max(\underline{m}, \underline{m}') \mid \underline{m} \in M, \underline{m}' \in M'\},$$
$$\max(M) := \max(M, M),$$

$$\min(M, M') := \{\min(\underline{m}, \underline{m}') \mid \underline{m} \in M, \underline{m}' \in M'\},$$
$$\min(M) := \min(M, M).$$

Definition 2.6. For each $\underline{m} \in \mathbb{N}^k_\omega$ let $\text{reg}(\underline{m}) := \{\underline{m}' \in \mathbb{N}^k \mid \underline{m}' \leq \underline{m}\}$ be the *region* specified by $\underline{m}$ and $\text{hyp}(\underline{m}) := \{\underline{m}' \in \mathbb{N}^k \mid \underline{m}(i) \neq \omega \text{ implies } \underline{m}'(i) = \underline{m}(i)\}$ denotes the *hyperplane* specified by $\underline{m}$ and restricted to $\mathbb{N}^k$.

Lemma 2.7. *For each* $\underline{m} \in \mathbb{N}^k_\omega$ *the sets* $\text{reg}(\underline{m})$ *and* $\text{hyp}(\underline{m})$ *are semilinear.*

Proof. Trivial and omitted.

Definition 2.8. A set $K \subseteq \mathbb{N}^k$ is called *right-closed* iff $K = K + \mathbb{N}^k$.

Definition 2.9. Let K be a subset of $\mathbb{N}^k$ then the *residue set* of K, written $\text{res}(K)$, is the smallest subset of K which satisfies $\text{res}(K) + \mathbb{N}^k = K + \mathbb{N}^k$.

By this definition $\text{res}(K)$ is a set of incomparable vectors with respect to the partial order $\leqq$ and therefore by Dicksons lemma finite. Thus we obviously have:

Lemma 2.10. *For each right-closed set* $K \subseteq \mathbb{N}^k$ $\text{res}(K)$ *is finite and* $K = \text{res}(K) + \mathbb{N}^k$ *is a representation of* K *as a semilinear set.*

Lemma 2.11. *If* K, K' *are right-closed sets, then* $K \cup K'$ *and* $K \cap K'$ *are right closed, too.*

Proof. Trivial and omitted.

If one knows the residue sets of the right-closed sets K and K', then it is easy to compute the sets $\text{res}(K \cup K')$ and $\text{res}(K \cap K')$.

Lemma 2.12. *Let* $K, K' \subseteq \mathbb{N}^k$ *be right-closed sets.*

(a) $\text{res}(K \cup K') = (\text{res}(K) \setminus K') \cup (\text{res}(K') \setminus K) \cup (\text{res}(K) \cap \text{res}(K'))$

(b) $\text{res}(K \cap K') = \text{res}(\max(\text{res}(K), \text{res}(K')))$.

Proof. Statement (a) is easily proved and thus left for the reader. To verify (b) observe that for any set $R \subseteq \mathbb{N}^k$ $\text{res}(R)$ is a set of incomparable vectors, and moreover for any $\underline{m}_0 \in K \cap K'$ there exist $\underline{m} \in \text{res}(K)$, $\underline{m}' \in \text{res}(K')$ with $\underline{m}_0 \geqq \max(\underline{m}, \underline{m}')$. Hence $\underline{m}_0 \geqq \underline{m}''$ for some $\underline{m}'' \in \text{res}(\max(\text{res}(K), \text{res}(K')))$.

If a right-closed set K is given in the form $K = L + \mathbb{N}^k$, then it is not always possible to effectively compute $\text{res}(K)$ from a finite representation of L. The next result exhibits a necessary and sufficient condition, called property RES, to effectively construct the finite set $\text{res}(K)$.

Definition 2.13. For each set $K \subseteq \mathbb{N}^k$ define the predicate $p_K: \mathbb{N}^k_\omega \to \{\text{true, false}\}$ by $p_K(\underline{m}) := (\text{reg}(\underline{m}) \cap K \neq \emptyset)$. A set K is said to have *property RES* iff the predicate $p_K(\underline{m})$ is decidable for each $\underline{m} \in \mathbb{N}^k_\omega$.

The following theorem is similar to Lemma 3 in [12], where an algorithm to compute $\text{res}(K)$ for a right closed set K is given. However the algorithm is not very practical, since it works by mere enumeration of the both sets K and $\mathbb{N}^k - K$ using the fact that membership is decidable for both sets. Here we want to use the property RES and a different algorithm which might have smaller complexity. However, as regards its complexity we can only give a lower bound.

Theorem 2.14. *Let* $K \subseteq \mathbb{N}^k$ *be a right-closed set. Then* $\operatorname{res}(K)$ *can be effectively constructed iff* K *has property RES.*

Proof. Assume first that $\operatorname{res}(K)$ can be computed. Then $K = \operatorname{res}(K) + \mathbb{N}^k$ gives a semilinear representation of K. Since $\operatorname{reg}(\underline{m})$ is a semilinear set, a representation of which can be found effectively, the question "$\operatorname{reg}(\underline{m}) \cap K = \emptyset$?" is decidable.

Conversely assume that the question "$\operatorname{reg}(\underline{m}) \cap K = \emptyset$?" is decidable for each $\underline{m} \in \mathbb{N}^k_\omega$. The following method can be used to effectively construct $\operatorname{res}(K)$:

Let K be a right-closed subset of $\mathbb{N}^k$ for which property RES holds, i.e. $p_K(\underline{m}) := (\operatorname{reg}(\underline{m}) \cap K \neq \emptyset)$ is decidable for each $\underline{m} \in \mathbb{N}^k$.

Algorithm to compute $\operatorname{res}(K)$

(1) **begin** (* initialization *)
(2) $i := 0$; $M_0 := \{(\omega, \ldots, \omega)\}$; $R_0 := \emptyset$;
(3) **repeat**
(4) choose some $\underline{m} \in M_i$;
(5) **if** $p_K(\underline{m}) = \text{false}$ **then** $M_i := M_i - \{\underline{m}\}$;
(6) **until**
(7) $p_K(\underline{m}) = \mathit{true}$ **or** $M_i = \emptyset$
(8) **endrepeat**;
(9) **if** $M_i = \emptyset$ **then** $\operatorname{res}(K) := R_i$ **and stop**
(10) **else**
(11) **begin** (* now $\operatorname{reg}(\underline{m}) \cap K \neq \emptyset$ and hence $\operatorname{reg}(\underline{m})$ contains at least one element of $\operatorname{res}(K)$; one such element will be found in the next repeat loop *)
(12) **repeat**
(13) choose some coordinate $\underline{m}(i)$ of $\underline{m}$ which in this loop has not been considered yet;
(14) replace $\underline{m}(i)$ in $\underline{m}$ by the smallest $n \in \mathbb{N}$ such that $p_K(\underline{m})$ for this new vector is still true;
(15) **until**
(16) all coordinates have been considered
(17) **endrepeat**; (* the new vector $\underline{m} \in \mathbb{N}^k$ found in this way will be an element of $\operatorname{res}(K)$ as will be shown in Lemma 2.15 below *)
(18) $R_{i+1} := R_i \cup \{\underline{m}\}$;

Let $\underline{m} = (x_1, \ldots, x_k)$ be the vector found in the preceding steps (lines (13) to (17)).

(19) $$M_i' := \left\{ (y_1, \ldots, y_k) \in \mathbb{N}^k_\omega \,\middle|\, \begin{array}{ll} \exists 1 \leq j \leq k\colon y_j := x_j - 1 & \text{and} \\ y_m = \omega & \text{for all } m \neq j \end{array} \right\}$$

(* M_i' is describing *all* the regions that do not contain the element $\underline{m}$, i.e. for $\operatorname{reg}(M_i') := \bigcup_{\underline{m}' \in M_i'} \operatorname{reg}(\underline{m}')$ one has $\mathbb{N}^k - \operatorname{reg}(M_i') = \{\underline{m}\} + \mathbb{N}^k$ *)

(20) $M_{i+1} := \min(M_i, M_i')$
(21) $i := i + 1$;

(22) **endif**
(23) **goto** line (3)
(24) **end** (* algorithm *).

Lemma 2.15. *The vector* $\underline{m}$ *constructed in lines* (13) *to* (17) *of the previous algorithm is an element of* res(K).

Proof. First of all, this new vector $\underline{m}$ is an element of the set K, since $p_K(\underline{m})$ = true. Now, for the sake of contradiction assume that $\underline{m} \notin \text{res}(K)$. Then there exists some $\underline{m}' \lneqq \underline{m}$ with $p_K(\underline{m}')$ = true, and, according to the sequence of choices made in the repeat loop to construct $\underline{m}$, there will be some coordinate $\underline{m}(j)$, which for *the first time* is larger than the corresponding coordinate $\underline{m}'(j)$. But this contradicts the fact that the coordinate $\underline{m}(j)$ was chosen to be the *smallest* $n \in \mathbb{N}$ such that the new vector $\underline{m}$ obtained in this step still satisfies $p_K(\underline{m})$.

Lemma 2.16. *At line* (3) *of the algorithm one always has:*
(a) $\text{reg}(M_i) \cap R_i = \emptyset$ *and*
(b) $\text{reg}(M_i) \supseteq \text{res}(K) - R_i$.

Proof. We shall use induction on i:

Basis
Obviously $\text{reg}(M_0) = \mathbb{N}^k$ and $R_0 = \emptyset$ so that (a) and (b) are satisfied trivially.

Induction Step
Assume $\text{reg}(M_i) \cap R_i = \emptyset$ and $\text{reg}(M_i) \supseteq \text{res}(K) - R_i$ at line (3). Then this remains true by going from line (3) to line (20) without passing through line (22), since in the repeat loop line (3) to line (8) only those $\underline{m} \in \mathbb{N}_\omega^k$ are substracted from M_i, for which $\text{reg}(\underline{m}) \cap \text{res}(K) = \emptyset$.

Let $\underline{m} \in \text{res}(K) - R_i$ be the new element computed to define $R_{i+1} := R_i \cup \{\underline{m}\}$ in line (18). Then M_i' computed in line (19) has the properties:
(a') $\underline{m} \notin \text{reg}(M_i')$ and
(b') $\underline{m}' \notin \text{reg}(M_i')$ implies $\underline{m} \leq \underline{m}'$ which is equivalent to

$$\mathbb{N}^k - \text{reg}(M_i') = \{\underline{m}\} + \mathbb{N}^k.$$

Now $\text{reg}(M_{i+1}) = \text{reg}(M_i) \cap \text{reg}(M_i')$ by definition of M_{i+1} and the property $\text{reg}(\min(\underline{x}, \underline{y})) = \text{reg}(\underline{x}) \cap \text{reg}(\underline{y})$.

Hence: $\text{reg}(M_{i+1}) \cap R_{i+1}$

$$= \text{reg}(M_i) \cap \text{reg}(M_i') \cap R_{i+1}$$
$$= \text{reg}(M_i) \cap \text{reg}(M_i') \cap (R_i \cup \{\underline{m}\})$$
$$= (\text{reg}(M_i) \cap \text{reg}(M_i') \cap R_i) \cup (\text{reg}(M_i) \cap \text{reg}(M_i') \cap \{\underline{m}\})$$

$= \emptyset$ since by induction $\text{reg}(M_i) \cap R_i = \emptyset$, and by construction of M_i' one has $\underline{m} \notin \text{reg}(M_i')$.

Thus, reaching line (3) again after having executed $i := i+1$ in line (21) property (a) is again true.

For showing the other property (b) observe that $\text{res}(K)-R_{i+1}=res(K)-R_i-\{\underline{m}\}=(\text{res}(K)-R_i)\cap(\text{res}(K)-\{\underline{m}\})$, since $R_{i+1}=R_i\cup\{\underline{m}\}$. Now $\text{res}(K)-R_i\subseteq\text{reg}(M_i)$ by induction and $\text{res}(K)-\{\underline{m}\}\subseteq\text{reg}(M_i')$ by properties (a′) and (b′) and the observation that $\underline{m}'$ and $\underline{m}$ are incomparable.

Hence $\text{res}(K)-R_{i+1}=(\text{res}(K)-R_i)\cap(\text{res}(K)-\{\underline{m}\})\subseteq\text{reg}(M_i)\cap\text{reg}(M_i')=\text{reg}(M_{i+1})$ as desired. This proves Lemma 2.16.

It is now easy to verify the total correctness of the algorithm as follows: Each time, that the statement in line (18): $R_{i+1}:=R_i\cup\{\underline{m}\}$ is executed, one has $\underline{m}\in\text{res}(K)$ (see Lemma 2.15) so that $R_i\subsetneqq R_{i+1}\subseteq\text{res}(K)$ for each i for which this statement is executed. Since by Lemma 2.16 we have $\text{reg}(M_{i+1})\supseteq\text{res}(K)-R_{i+1}$, the first repeat loop, (lines (3) to (8)), will always find a new element in $\text{reg}(M_{i+1})\cap(\text{res}(K)-R_{i+1})$ if it exists. Since $\text{res}(K)$ is finite there will be some index j such that $R_j=\text{res}(K)$ and then $M_j\cap\text{res}(K)=\emptyset$ which implies $p_K(\underline{m})=\text{false}$ for each $\underline{m}\in M_j$ and hence the algorithm will correctly terminate by emptying M_j at the stop statement in line (9) with final output $\text{res}(K):=R_j$.

3. Computing Certain Right-Closed Sets in Petri Nets

Let us first fix some notation for Petri nets or more precisely P/T-nets. For much more detail see [17].

Definition 3.1. A P/T-net $N=(P,T,F,B)$ is defined by

a finite set P of *places*,

a finite set T of *transitions*, disjoint from P, and two mappings:

$$F: P\times T\to\mathbb{N}$$
$$B: P\times T\to\mathbb{N}$$

called *forward* and *backward incidence mapping*. They can also be seen as $(/P/,/T/)$-matrices over $\mathbb{N}$, (where $/S/$ is the cardinality of a set S). Let $\Delta:=B-F$ be the *incidence matrix* of the P/T-net N. $F(t)$, $B(t)$ and $\Delta(t)$ denote the t-column vector in $\mathbb{N}^{/P/}$ of F, B and Δ, respectively.

Definition 3.2. A marking $\underline{m}\in\mathbb{N}^{/P/}$ is a column vector giving a number $\underline{m}(p)$ of tokens for each place $p\in P$. A transition *has concession* in $\underline{m}$, written $\underline{m}(t\rangle$, iff $F(t)\leq\underline{m}$. For $\underline{m}\in\mathbb{N}_\omega^{/P/}$ we also write $\underline{m}(t\rangle$ iff $\exists\,\underline{m}'\in\text{reg}(\underline{m})$: $\underline{m}'(t\rangle$.

For $\underline{m}\in\mathbb{N}_\omega^{/P/}$ we define $\underline{m}(t\rangle\underline{m}'$ iff $m(t\rangle$ and $\underline{m}'=\underline{m}+B(t)-F(t)=\underline{m}+\Delta(t)$. We extend this notion to strings $w\in T^*$ by

(a) $\underline{m}(\lambda\rangle\underline{m}$ for all $\underline{m}\in\mathbb{N}_\omega^{/P/}$ and

(b) $\underline{m}(wt\rangle m''$ iff $\exists\,\underline{m}'\in\mathbb{N}_\omega^{/P/}$: $\underline{m}(w\rangle m'$ and $\underline{m}'(t\rangle\underline{m}''$.

Again we say that w *has concession* in $\underline{m}\in\mathbb{N}_\omega^{/P/}$, written $\underline{m}(w\rangle$, iff $\exists\,\underline{m}'\in\mathbb{N}_\omega^{/P/}$: $\underline{m}(w\rangle\underline{m}'$.

For $\underline{m}\in\mathbb{N}_\omega^{/P/}$ we let $\Omega(\underline{m}):=\{p\in P\mid\underline{m}(p)=\omega\}$.

Definition 3.3. A P/T-net $N=(P,T,F,B)$ together with an initial marking $\underline{m}_0\in\mathbb{N}^{/P/}$ and/or a labelling homomorphism h: $T^*\to X^*$ will be also called a *P/T-net* and is denoted by $(N,\underline{m}_0)$ and $(N,h,\underline{m}_0)$, respectively. For such a P/T-

net $(N, \underline{m}_0)$ and a subset $K \subseteq \mathbb{N}^{|P|}$ we define the *K-restricted set of firing sequences*

$$F_K(N, \underline{m}_0) := \left\{ t_{i_1} t_{i_2} \ldots t_{i_n} \in T^+ \;\middle|\; \begin{array}{l} \underline{m}_0 (t_{i_1}\rangle \underline{m}_1 (t_{i_2}\rangle \underline{m}_2 \ldots \underline{m}_{n-1} (t_{i_n}\rangle \underline{m}_n \\ \text{for markings } \underline{m}_i \in K \quad (0 \leq i \leq n) \end{array} \right\}$$

and the *K-restricted reachability set*

$$R_K(N, \underline{m}_0) := \{ \underline{m} \in \mathbb{N}^{|P|} \mid \exists\, w \in F_K(N, \underline{m}_0) : \underline{m}_0 (w\rangle \underline{m} \}.$$

For $K = \mathbb{N}^{|P|}$ these sets are the ordinary *set of firing sequences* $F(N, \underline{m}_0)$ and the *reachability set* $R(N, \underline{m}_0)$, respectively.

For a net $(N, h, \underline{m}_0)$ the *language* is defined by $L(N, h, \underline{m}_0) := \{ h(w) \mid w \in F(N, \underline{m}_0) \}$. Until section 5 we assume $h(t) \neq \lambda \; \forall\, t \in T$.

Definition 3.4. Let $\Delta: T^* \to \mathbb{Z}^{|P|}$ be a homomorphism defined as follows:

$$\Delta(\lambda) := \underline{0} \text{ (null-vector of suitable dimension)}$$

$$\Delta(t) := B(t) - F(t), \text{ and } \Delta(uv) := \Delta(u) + \Delta(v) \quad \text{for} \quad u, v \in T^*$$

We also use the *Parikh image* $\Psi: T^* \to \mathbb{N}^{|T|}$, where $\Psi(w)(t)$ gives the number of occurences of the transition t in the finite word $w \in T^*$. We will also write $\Psi(w)(t) = \omega$ if w is an infinite sequence and this number is not finite. Δ and Ψ are related as follows

$$\Delta(w) = \Delta \cdot \Psi(w)$$

which motivates the choice of the same symbol Δ for both notions (homomorphism and incidence matrix).

Modelling concurrent systems by Petri nets also the infinite behaviour is of importance. In this paper we also use the notion of infinite firing sequence of a P/T-net [28].

Definition 3.5. X^ω denotes the set of *infinite words* $w = w(1)\, w(2) \ldots$ over the alphabet X. For $i \in \mathbb{N}$ $w(i)$ denotes the i-th element of w and $w[i] = w(1)\, w(2) \ldots w(i)$ the prefix of length i of w.

For $w \in X^\omega$ the set $\mathrm{In}(w) := \{ x \in X \mid x = w(i) \text{ for infinitely many } i \in \mathbb{N} \}$ is called *infinity set* of w.

An ω-word $w \in T^\omega$ of transitions in a net $N = (P, T, F, B)$ is said to *have concession* in a marking $\underline{m} \in \mathbb{N}^{|P|}$, again written $\underline{m}(w\rangle$, if $\underline{m}(w[i]\rangle$ for all $i \in \mathbb{N}$. $F_\omega(N, \underline{m}_0) := \{ w \in T^\omega \mid \underline{m}_0 (w\rangle \}$ is the set of all *infinite firing sequences* of N with initial marking $\underline{m}_0$.

For a motivated introduction to place/transition nets we refer to [17] and [29], where also the following construction of the coverability graph is used. It differs in some way from the original form in [19]. The most important difference used here, is the possibility to start with an initial node containing ω-coordinates.

Definition 3.6. Let $N = (P, T, F, B)$ be a P/T-set and $\underline{m}_0 \in \mathbb{N}_\omega^{|P|}$. A *coverability graph* $G(N, \underline{m}_0)$ of N will be a finite, directed, edge labelled graph consisting of a set of nodes $\text{NODES} \subseteq \mathbb{N}_\omega^{|P|}$, and a set $\rightarrow \subseteq \text{NODES} \times T \times \text{NODES}$ of labelled arcs. $G(N, \underline{m}_0)$ is defined by the following construction:

```
(1)  begin
(2)  NODES := {m_0}; → := ∅;
(3)  loop
(4)  choose m ∈ NODES, t ∈ T such that F(t) ≦ m and the pair (m, t) has not
     been considered before;
(5)  if no such pair (m, t) exists then stop fi;
(6)  m' := m − F(t) + B(t);
(7)  if m' ∈ NODES
(8)  then
(9)  → := → ∪ {(m, t, m')};
(10) else begin
(11) m'' := m' + ω ·          Σ            (m' − m''')
                   (m''' -*-> m) ∧
                   (m''' ≦ m')
(12) NODES := NODES ∪ {m''}; → := → ∪ {(m, t, m'')}
(13) end if;
(14) goto line (3)
(15) end (* of construction *)
```

It is well known that the construction of the coverability graph always terminates, however, the resulting graph is usually not unique, because different choices at the beginning of the loop can produce different graphs.

Definition 3.7. Let $G := G(N, \underline{m}_0)$ be some coverability graph. For each node $\underline{m}' \in \text{NODES}$ of G define $L(G, \underline{m}') := \{v \in T^* \mid \underline{m}' \xrightarrow[v]{*} \underline{m}'' \text{ is a path in } G\}$

and
$$L(G) := \bigcup_{\underline{m}' \in \text{NODES}} L(G, \underline{m}')$$

Lemma 3.8. *Let $G(N, \underline{m}_0)$ be some coverability graph. Then $L(G)$ and $L(G, \underline{m}')$ for each $\underline{m}' \in NODES$ are regular subsets of T^* and effectively constructable from G. For each $\underline{m}'' \in \text{reg}(\underline{m}')$ the* set *$F(N, \underline{m}'')$ is a subset of $L(G, \underline{m}')$. In addition, a set of places $P' \subseteq P$ is simultaneously unbounded in $R(N, \underline{m}_0)$ iff $\exists \underline{m} \in NODES: \Omega(\underline{m}) = P'$.*

Proof. First of all, as is well known, $L(G)$ and $L(G, \underline{m}')$ are regular sets since the coverability graph $G(N, \underline{m}_0)$ is finite. Clearly $F(N, \underline{m}'') \subseteq L(G, \underline{m}')$ for each $\underline{m}'' \in \text{reg}(\underline{m}')$, since then we have $\underline{m}'' \leqq \underline{m}$.

The last statement about the set $P' \subseteq P$ of simultaneously unbounded places is Theorem 3.11 in [13].

Lemma 3.9. *Let $N = (P, T, F, B)$ and $G(N, \underline{m}_0)$ be some coverability graph of N with initial node $\underline{m}_0 \in \mathbb{N}_\omega^{|P|}$. Then $v \in L(G)$ and $\Delta(v) \geqq 0$ implies $\exists u \in T^* \, \exists \underline{m}' \in \text{reg}(\underline{m}_0): uv \in F(N, \underline{m}')$.*

Proof. We first quote Theorem 1 (b) from [29], which should be also clear from Lemma 3.8 or [19]:

If $\underline{m}_0 \in \mathbb{N}^{|P|}$ is an initial marking of N and $\underline{m}$ some node in $G(N, \underline{m}_0)$, then for every $k \in \mathbb{N}$ there is some firing sequence $u_k \in F(N, \underline{m}_0)$ with $\underline{m}_0(u_k\rangle\, \underline{m}_k$, such that $\underline{m}_k(p) \geqq k$ for all $p \in \Omega(\underline{m})$ and $\underline{m}_k(p) = \underline{m}(p)$ for all $p \notin \Omega(\underline{m})$.

If we replace $\underline{m}_0 \in \mathbb{N}^{|P|}$ by $\underline{m}_0 \in \mathbb{N}_\omega^{|P|}$ as required in the Lemma, then the claim remains true if the initial marking of N is replaced by some suffiently large $\underline{m}' \in \text{reg}(\underline{m}_0)$.

From this remark the Lemma is proved as follows. If $v \in L(G)$ and $\Delta(v) \geqq 0$ then there is a path $\underline{m}_1 \xrightarrow{v} \underline{m}_2$ in $G(N, \underline{m}_0)$. If $\Omega(\underline{m}_1) \neq \Omega(\underline{m}_2)$ then by $\Delta(v) \geqq 0$ and $\underline{m}_2 \geqq \underline{m}_1$ we have $\underline{m}_2 \xrightarrow{v} \underline{m}_3$ with $\Omega(\underline{m}_2) = \Omega(\underline{m}_3)$ for a third node $\underline{m}_3$. Since the set of places is finite, after a finite number of repetitions of this step, we reach nodes $\underline{m}_i$ and $\underline{m}_j$ such that $\underline{m}_i \xrightarrow{v} \underline{m}_j$ in $G(N, \underline{m}_0)$ and $\Omega(\underline{m}_i) = \Omega(\underline{m}_j)$. Now v can be fired in the net from every marking $\underline{m}_q \in \mathbb{N}^{|P|}$ with $\underline{m}_q(p) = \underline{m}_i(p)$ for all $p \notin \Omega(\underline{m}_i)$ and a sufficiently large number k of tokens in all places $p \in \Omega(\underline{m}_i)$. By the claim mentioned in the beginning of the proof there is $\underline{m}' \in \text{reg}(\underline{m}_0)$ and $u_k \in T^*$, such that $\underline{m}_q$ with $\underline{m}'(u_k\rangle \underline{m}_q$ is such a marking. By $u_k v \in F(N, \underline{m}')$ we have shown the Lemma.

Definition 3.10. Let $N = (P, T, F, B)$ be a fixed P/T-net and $\underline{m} \in \mathbb{N}^{|P|}$ be an arbitrary marking of N.

(a) $\underline{m}$ is *$\hat{T}$-blocked* for a set $\hat{T} \subseteq T$ of transitions iff no transition $t \in \hat{T}$ has concession in a reachable marking $\underline{m}' \in R(N, \underline{m})$. When $\hat{T} = T$ then $\underline{m}$ is a *total deadlock*. (For $T = \{t\}$ $\underline{m}$ is often called t-dead which we want to avoid because of possible confusion with the next definition.)

(b) $\underline{m}$ is called *dead*, iff $F(N, \underline{m})$ is finite.

Remark: If $\underline{m}$ is dead, then total deadlocks cannot be avoided. Such situations are sometimes called unsafe.

(c) $\underline{m}$ is called *bounded*, iff $R(N, \underline{m})$ if finite. Otherwise $\underline{m}$ is called *unbounded*.

(d) $\underline{m}$ is called *$\hat{T}$-continual* for some subset $\hat{T} \subseteq T$ of transitions, iff there is some infinite string $w \in T^\omega$ such that $\underline{m}(w\rangle$ and $\hat{T} \subseteq \text{In}(w)$.

Remark: Every live marking $\underline{m}$ is $\hat{T}$-continual for $\hat{T} = T$, but the converse is usually not true. A marking $\underline{m}$ is $\hat{T}$-continual iff the predicate hot $(\hat{T}, \underline{m})$ in [18] is true.

Now we define the following sets of markings according with (a) to (d) above:

$$\text{(aa)}\ \ \text{NOTBLOCKED}(\hat{T}) := \{\underline{m} \in \mathbb{N}^{|P|} \mid \underline{m} \text{ is not } \hat{T}\text{-blocked}\}$$

$$\text{(bb)}\ \ \text{NOTDEAD} := \{\underline{m} \in \mathbb{N}^{|P|} \mid \underline{m} \text{ is not dead}\}$$

$$\text{(cc)}\ \ \text{UNBOUNDED} := \{\underline{m} \in \mathbb{N}^{|P|} \mid \underline{m} \text{ is unbounded}\}$$

$$\text{(dd)}\ \ \text{CONTINUAL}(\hat{T}) := \{\underline{m} \in \mathbb{N}^{|P|} \mid \underline{m} \text{ is } \hat{T}\text{-continual}\}$$

From the monotonicity property of Petri nets it follows immediately that the four sets of markings defined by (aa) to (bb) are all right-closed. We shall now show that they also satisfy property RES.

Theorem 3.11. *Let $N=(P,T,F,B)$ be a fixed net and $\hat{T}\subseteq T$ be arbitrary. Then each set $K\in\{NOTBLOCKED(\hat{T}), NOTDEAD, UNBOUNDED, CONTINUAL(\hat{T})\}$ satisfies property RES.*

Proof. Let be $G:=G(N,\underline{m})$ for some $\underline{m}\in\mathbb{N}^{/P/}$.

Case 1: $K=$ NOTBLOCKED$(\hat{T})$
From Lemma 3.8 one concludes that $\text{reg}(\underline{m})\cap K\neq\emptyset$ iff for some $t\in\hat{T}$ there exists an arc in G which is labelled by t, i.e. $t\in L(G)$. This clearly is decidable, hence the set K has property RES.

Case 2: $K=$ NOTDEAD
Again from Lemma 3.8 one concludes that $\text{reg}(\underline{m})\cap K\neq\emptyset$ iff there exists $v\in L(G)$ such that $\Delta(v)\geqq\underline{0}$. Since $L(G)$ is a regular subset of T^*, which can be constructed from G effectively, the set $\Delta(L(G)):=\{\Delta(v)|v\in L(G)\}$ is a regular, hence semilinear, subset of $\mathbb{Z}^{/P/}$. Then $S:=\Delta(L(G))\cap\mathbb{N}^{/P/}$ is a semilinear subset of $\mathbb{N}^{/P/}$, a representation of which can be constructed effectively. Hence "$S\neq\emptyset$" is decidable and $S\neq\emptyset$ iff $\exists v\in L(G)$: $\Delta(v)\geqq\underline{0}$. Thus K has property RES.

Case 3: $K:=$ UNBOUNDED
Again we find, using Lemma 3.8, $\text{reg}(\underline{m})\cap K\neq\emptyset$ iff $\exists v\in L(G)$: $\Delta(v)\geqq\underline{0}$ (with $\Delta(v)\neq\underline{0}$). We construct the semilinear set $S:=\Delta(L(G))\cap\{\underline{m}'\in\mathbb{N}^{/P/}|\underline{m}'\neq\underline{0}\}$ and then $S\neq\emptyset$ iff $\text{reg}(\underline{m})\cap K\neq\emptyset$, which is decidable using the finite representation of S. Hence, also in this case the set K has property RES.

Case 4: $K=$ CONTINUAL$(\hat{T})$
Let $\underline{e}_{\hat{T}}\in\mathbb{N}^{/T/}$ be defined by $\underline{e}_{\hat{T}}(t):=$ **if** $t\in\hat{T}$ **then** 1 **else** 0 **fi**. We first show the following claim:

Claim: $\text{reg}(\underline{m})\cap K\neq\emptyset$ iff $\exists v\in L(G)$: $\Delta(v)\geqq\underline{0}$ and $\Psi(v)\geqq\Psi(\underline{e}_{\hat{T}})$. To see this assume first, that there exists $v\in L(G)$ such that $\Delta(v)\geqq\underline{0}$ and $\Psi(v)\geqq\Psi(\underline{e}_{\hat{T}})$. Then by Lemma 3.9 there exists $\underline{m}'\in\text{reg}(\underline{m})$ and a string $u\in T^*$ such that $\underline{m}'(uv\rangle$. Since $\Delta(v)\geqq\underline{0}$ also $\underline{m}'(uv^n\rangle$ for every $n\in\mathbb{N}$ and $\underline{m}'$ is $\hat{T}$-continual by $\hat{T}\subseteq\text{In}(v^\omega)$.

Conversely, if $m'\in\text{reg}(\underline{m})$ is $\hat{T}$-continual, then there exists an infinite sequence $w\in T^\omega$, such that $\underline{m}'(w\rangle$ and $\hat{T}\subseteq\text{In}(w)$. Obviously w has a decomposition $w=w_1w_2w_3\ldots$, where $w_i\in T^*$ and $\Psi(w_i)\geqq\Psi(\underline{e}_{\hat{T}})$.

Now $\underline{m}'(w_1\rangle\underline{m}_1$, $\underline{m}'(w_1w_2\rangle\underline{m}_2$, $\underline{m}'(w_1w_2w_3\rangle\underline{m}_3,\ldots$ defines an infinite sequence of markings $\underline{m}'$, $\underline{m}_1$, $\underline{m}_2,\ldots$. Therefore there must exist indices $i<j$ such that $\underline{m}_i\leqq\underline{m}_j$.

Defining $v:=w_{i+1}w_{i+2}\ldots w_j$ we then have $\underline{m}_i(v\rangle\underline{m}_j$ with $\Delta(v)\geqq\underline{0}$ and $\Psi(v)\geqq\Psi(\underline{e}_{\hat{T}})$.

Since $\underline{m}_i\in R(N,\underline{m}')$ there exists $u\in T^*$ such that $\underline{m}'(u\rangle\underline{m}''$ with $\underline{m}_i\in\text{reg}(\underline{m}'')$, hence $uv\in L(G,\underline{m})$ and $v\in L(G)$. This proves the claim.

Now, in order to decide whether there exists some $v\in L(G)$ with $\Delta(v)\geqq\underline{0}$ and $\Psi(v)\geqq\Psi(\underline{e}_{\hat{T}})$ we proceed as follows:

First, $R:=L(G)\cap\{w\in T^*|\Psi(w)\geqq\Psi(\underline{e}_{\hat{T}})\}$ is a regular set, since it is the intersection of two regular sets. A finite representation of R can be constructed from the coverability graph $G=G(N,\underline{m})$. Then $S:=\Delta(R)\cap\mathbb{N}^k$ is a semilinear set, a finite representation of which can be effectively constructed.

The question "$S \neq \emptyset$?" is therefore decidable and equivalent to:

$$\text{"}\exists\, v \in L(G)\colon \Delta(v) \geqq \underline{0} \wedge \Psi(v) \geqq \Psi(\underline{e}_{\hat{T}})\text{?"}$$

Hence also in this case the set K has property RES.

The following result is a direct consequence of the proof of Theorem 3.11 and solves problem P24 (iii) of the problem collection in [9].

Theorem 3.12. *Given a P/T-net $N=(P,T,F,B)$, a marking $\underline{m} \in \mathbb{N}^{|P|}$, and a set $\hat{T} \subseteq T$ of transitions, then*

(a) *It is decidable, whether $\underline{m}$ is $\hat{T}$-continual.*

(b) *It is decidable, whether $\underline{m}$ is $\hat{T}$-blocked.*

(c) *It is decidable, whether there exists an infinite firing sequence $w \in T^{\omega}$ such that $m(w\rangle$ and $In(w)=\hat{T}$.*

Proof. The claim in Case 4 of the proof for Theorem 3.11 says, that $\underline{m}$ is $\hat{T}$-continual iff some coverability graph $G(N,\underline{m})$ contains a path $\underline{m}' \xrightarrow[v]{*} \underline{m}''$, labelled by $v \in T^*$, such that $\Delta(v) \geqq \underline{0}$ and each $t \in \hat{T}$ occurs at least once in v. Hence we have (a).

Part (b) is even more simple, since Case 1 of the preceding proof says, that $\underline{m}$ is $\hat{T}$-blocked iff $G(N,\underline{m})$ does not contain an arc $\underline{m}' \xrightarrow[t]{} \underline{m}''$ labelled by some $t \in \hat{T}$.

From the arguments given to verify the claim in Case 4 of Theorem 3.11 one easily deduces that $\underline{m}$ has the desired property of (c) iff $G(N,\underline{m})$ contains a path $\underline{m}' \xrightarrow[v]{*} \underline{m}''$ such that $v \in \hat{T}^*$, $\Delta(v) \geqq \underline{0}$ and each $t \in \hat{T}$ occurs at least once within v.

The main result of this section can now be stated as follows:

Theorem 3.13. *For each $K \in \{NOTBLOCKED(\hat{T})$; $NOTDEAD$; $UNBOUNDED$; $CONTINUAL(\hat{T})\}$ the finite set* res(K) *can be constructed effectively.*

Proof. Immediate consequence of Theorem 3.11 and Theorem 2.14.

An important application of Theorem 3.13 concerns the question, whether a given P/T-net is bounded for every initial marking.

Definition 3.14. A P/T-net $N=(P,T,F,B)$ is called *bounded*, iff $R(N,\underline{m})$ is finite for each marking $\underline{m} \in \mathbb{N}^{|P|}$.

Theorem 3.15. *It is decidable, whether a given P/T-net $N=(P,T,F,B)$ is bounded.*

Proof. N is bounded iff res(UNBOUNDED)$=\emptyset$, which is decidable by Theorem 3.13.

This Theorem has been proved in [2] by a completely different method. Known results on the boundedness problem allow to give a hint concerning the complexity of the algorithms considered here. A marking $\underline{m}$ of a P/T-net N is bounded iff there is no $\underline{m}' \in$res(UNBOUNDED) with $\underline{m}' \leqq \underline{m}$. On the other hand, there is a constant c such that boundedness of a marking $\underline{m}$ in a P/T-net N cannot be decided in space $2^{c \cdot \sqrt{\text{size}(N)}}$ [21, 24]. The complexity of computing res(UNBOUNDED) cannot be smaller than this lower bound.

4. Controlling a *P/T*-Net Using Residue Sets

Having computed a residue res(K) of a right-closed set K, it may be useful to control a net in such a way that all reachable markings are lying in K. For the examples K=NOTDEAD and K=CONTINUAL(T) this is of particular importance, however, there will be other examples of interest too.

In the following we shall present a general construction for controlling the behaviour of an arbitrary P/T-net by some right-closed set K, just by changing its set of transitions and without adding new places.

Properties of controlled nets using particular right-closed sets will be considered in section 5.

Construction 4.1

Let $(N, \underline{m}_0)$ with $N=(P, T, F, B)$ be a P/T-net and $K \subseteq \mathbb{N}^{/P/}$ be a right-closed net satisfying property RES.

Then using the residue set res(K) we effectively construct the *K-restriction* $(N_K, h, \underline{m}_0)$, or $(N_K, \underline{m}_0)$ if h is not important, by a P/T-net $N_K=(P, T', F', B')$ and λ-free homomorphism h: $T'^* \to T^*$ as follows:

a) $T' := T_1 \cup T_2$
where $T_1 := \{t \in T \mid \forall \underline{m}' \in \text{res}(K)\ \exists \underline{m} \in \text{res}(K): \max(\underline{m}', F(t)) + \Delta(t) \geqq \underline{m}\}$
  and $T_2 := \{t_{\underline{m}} \mid t \in T - T_1, \underline{m} \in \text{res}(K)\}$

b) for all $t \in T_1$ let $F'(t) := F(t)$ and $B'(t) := B(t)$

c) for all $t_{\underline{m}} \in T_2$ let

$$F'(t_{\underline{m}}) := \max[F(t), \underline{m} - \Delta(t)]$$
$$B'(t_{\underline{m}}) := \max[B(t), \underline{m}]$$

(Recall that by Definition 2.5 max is evaluated for each place-component separately).

Since

$$F(t,p) \geqq \underline{m}(p) - \Delta(t,p) \Leftrightarrow$$
$$F(t,p) \geqq \underline{m}(p) - B(t,p) + F(t,p) \Leftrightarrow$$
$$B(t,p) \geqq \underline{m}(p)$$

part c) can be equivalently formulated by c′) for all $t_{\underline{m}} \in T_2$, $p \in P$ let

$$(F'(t_{\underline{m}}, p), B'(t_{\underline{m}}, p)) := \textbf{if } B(t,p) \geqq \underline{m}(p) \textbf{ then } (F(t,p), B(t,p)) \textbf{ else } (\underline{m}(p) - \Delta(t,p), \underline{m}(p))$$

d) h is defined by

$$h(t') := \begin{cases} t' & \text{if } t' \in T_1 \\ t & \text{if } t' = t_{\underline{m}} \in T_2 \end{cases}$$

If $(N, \underline{m}_0)$ is given together with initial marking $\underline{m}_0$, then the K-restriction $(N_K, h, \underline{m}_0)$ is defined only if $\underline{m}_0 \in K$.

Remark. Note, that even if t, $t' \in T$ are such that $F(t) \neq F(t')$ and $B(t) \neq B(t')$ and $\underline{m}, \underline{m}' \in \text{res}(K)$ are all different, then $F'(t_{\underline{m}}) = F'(t'_{\underline{m}'})$ and $B'(t_{\underline{m}}) = B'(t'_{\underline{m}'})$ is possible and only one of $t_{\underline{m}}$ and $t'_{\underline{m}'}$ is actually needed.

Also for $\underline{m} \neq \underline{m}'$, $\underline{m}, \underline{m}' \in \text{res}(K)$ it can happen that

$$F'(t_{\underline{m}}) \leqq F'(t_{\underline{m}'})$$

which means, that whenever $t_{\underline{m}'}$ is enabled in N_K so is $t_{\underline{m}}$ and since $\Delta'(t_{\underline{m}}) = \Delta'(t_{\underline{m}'})$ we can omit the transition $t_{\underline{m}'}$ and use $T' - \{t_{\underline{m}'}\}$ instead of T' without affecting the result.

We will show that N_K behaves like N, when only markings in K are used. To be more precise, the reachable markings of N, starting in $\underline{m}_0 \in K$ and never leaving K in between, are exactly those of N_K and a transition t in N can fire iff an equally labelled transition t' in N_K can do so.

Theorem 4.2. *Let N be a P/T-net, K a right-closed set, and N_K the K-restricted P/T-net from Construction 4.1.*

Then for all $\underline{m}_1 \in K$, $t \in T$ we have:

(a) $\underline{m}_1(t\rangle \underline{m}_2$ *in N and $\underline{m}_2 \in K$ iff*

$$\exists t' \in T': h(t') = t \wedge \underline{m}_1(t'\rangle \underline{m}_2 \quad \textit{in} \quad N_K.$$

In particular for all initial markings $\underline{m}_0 \in K$ we have

(b) $R(N_K, \underline{m}_0) = R_K(N, \underline{m}_0)$ *and* (c) $L(N_K, h, \underline{m}_0) = F_K(N, \underline{m}_0)$

Proof. We shall first show:

Claim 1
$\forall t' \in T': \Delta(h(t')) = \Delta(t')$

Proof of Claim 1. The claim is obvious for $t \in T_1$. Now let

$$t' = t_{\underline{m}} \in T_2 \quad \text{with} \quad t \in T - T_1, \quad \underline{m} \in \text{res}(K), \quad \text{and} \quad p \in P.$$

If $B(t,p) \geqq \underline{m}(p)$ then $\Delta(t',p) = B(t,p) - F(t,p) = \Delta(t,p)$, and if $B(t,p) < \underline{m}(p)$ then $\Delta(t',p) = B'(t',p) - F'(t',p) = \underline{m}(p) - (\underline{m}(p) - \Delta(t,p)) = \Delta(t,p)$, both by c') of construction 4.1.

Claim 2. For every $\underline{m}_1 \in K$: $\underline{m}_1(t\rangle \underline{m}_2$ in N and $\underline{m}_2 \in K$ implies $\underline{m}_1(t'\rangle \underline{m}_2$ in N_K for some $t' \in T'$ with $h(t') = t$.

Proof of Claim 2. If $t \in T_1$ take $t' := t$. If $t \in T - T_1$ take $t' := t_{\underline{m}}$ for some $\underline{m} \in \text{res}(K)$ with $\underline{m}_2 \geqq \underline{m}$. By Claim 1 it is sufficient to show

$$\underline{m}_1 \geqq F'(t').$$

Indeed $\underline{m}_1 = \underline{m}_2 - \Delta(t') = \underline{m}_2 - \Delta(t) \geqq \underline{m} - \Delta(t)$, and $\underline{m}_1 \geqq F(t)$, which implies $\underline{m}_1 \geqq \max[F(t), \underline{m} - \Delta(t)] = F'(t')$

Claim 3. For every $\underline{m}_1 \in K$: $\underline{m}_1(t'\rangle \underline{m}_2$ in N_K implies $\underline{m}_1(h(t')\rangle \underline{m}_2$ in N and $\underline{m}_2 \in K$.

Proof of Claim 3. If $t' \in T_1$ then $h(t') = t$. For $\underline{m}_1 \geqq \underline{m}'$ with $\underline{m}' \in \text{res}(K)$ by definition of T_1 there is $\underline{m} \in \text{res}(K)$ with $\max(\underline{m}', F(t)) + \Delta(t) \geqq \underline{m}$. Hence $\underline{m}_2 = \underline{m}_1 + \Delta(t) \geqq \max(\underline{m}', F(t)) + \Delta(t) \geqq \underline{m} \in K$. If $t' \in T - T_1$ then $t' = t_{\underline{m}}$ for some $\underline{m} \in \text{res}(K)$. Then $\underline{m}_1 \geqq F'(t_{\underline{m}}) = \max(\underline{m} - \Delta(t), F(t)) \geqq F(t)$ and $\underline{m}_2 = \underline{m}_1 - F'(t_{\underline{m}}) + B'(t_{\underline{m}}) \geqq B'(t_{\underline{m}}) = \max(B(t), \underline{m}) \geqq \underline{m} \in K$. Part a) of Theorem 4.2 now follows from claim 2 and claim 3. From this part b) and c) can be easily derived by induction on the reachability set.

Remark. It is important to note that, even though $R(N_K, \underline{m}_0) = R_K(N, \underline{m}_0)$, it is often the case that $R(N_K, \underline{m}_0) \neq R(N, \underline{m}_0) \cap K$.

To illustrate the construction of the K-restriction N_K and Theorem 4.2 we give the following example.

Example 4.3. Consider the P/T-net N in Fig. 4.1. a) and $K := \text{res}(K) + \mathbb{N}^4$ with $\text{res}(K) = \{\underline{m}_1, \underline{m}_2\}$ and $\underline{m}_1 := (2,0,0,0)$, $\underline{m}_2 := (0,0,0,1)$.

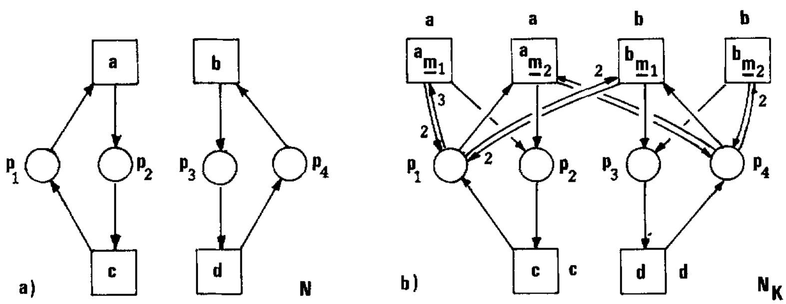

Fig. 4.1.

By $\Delta(c) = (1, -1, 0, 0)$ and $\Delta(d) = (0, 0, -1, 1)$ and following the notation of Construction 4.1 we obtain:

$$T_1 = \{c, d\}, \qquad T_2 = \{a_{\underline{m}_1}, a_{\underline{m}_2}, b_{\underline{m}_1}, b_{\underline{m}_2}\}$$

The construction of F and B results in the P/T-net N_K of Fig. 4.1 b). For each transition t the labelling $h(t)$ is given outside the box of t. To give an application of Theorem 4.2 we consider the initial marking $\underline{m}_0 := \underline{m}_1 + \underline{m}_2 \in K$.

Instead of looking at particular firing sequences we give an interpretation of N_K.

For the P/T-net $(N, \underline{m}_0)$ all reachable markings $\underline{m} \in R(N, \underline{m}_0)$ satisfy the following "invariant equations":

$$i_1: \ \underline{m}(p_1) + \underline{m}(p_2) = 2$$
$$i_2: \ \underline{m}(p_3) + \underline{m}(p_4) = 1$$

Together with property $R(N_K, \underline{m}_0) \subseteq K$ of N_K it follows

$$(*): \forall \underline{m} \in R(N_K, \underline{m}_0): (\underline{m}(p_2) = 0 \vee \underline{m}(p_3) = 0)$$

Hence, places p_2 and p_3 can be seen as "critical sections" of two "reader processes", represented by the two tokens in p_1 or p_2 and a "writer process" in the right hand side part of N_k. By Theorem 4.2 the labelled firing sequences $h(w) \in L(N_k, h, \underline{m}_0)$ are exactly those firing sequences $w \in F(N, \underline{m}_0)$ that respect the condition (*) of mutual exclusion. (How the net N_K can be systematically simplified will be shown later on).

Remark. In this example by the construction of the K-restriction we have found an "implementation" of the fact (*) in the sense of [11]. This observation can be generalized as follows:

Every fact with bounded input places can be
"implemented" by a K-restriction.

This is true, since facts can be equivalently formulated as

$$\forall \underline{m} \in R(N, \underline{m}_0): (\underline{m}(p_1) \geqq k_1) \vee \ldots \vee (\underline{m}(p_r) \geqq k_r)$$

where $\{p_1, \ldots, p_r\}$ are the output places of the fact together with the complementary places of the input places.

In the remainder of this section we give some methods how the net N_K can be simplified. This part can be skipped for a first reading.

The definition of N_K by Construction 4.1 is fairly general and independent from the initial marking $\underline{m}_0$. The only and obvious requirement is $\underline{m}_0 \in K$, because $\underline{m}_0 \notin K$ implies $R_K(N, \underline{m}_0) = F_K(N, \underline{m}_0) = \emptyset$ so that no construction would be needed.

This independence, however, usually leads to the construction of large P/T-nets N_K that could in many cases be simplified if there is just one fixed initial marking $\underline{m}_0$ for which the new P/T-net N_K has to be built.

For instance, the following case may occur: The initial marking $\underline{m}_0 \in K$ is $\{t\}$-blocked in $(N, \underline{m}_0)$ for some $t \in T$. Then transition t is not needed for the construction of $(N_K, \underline{m}_0)$ and can (and should) therefore be removed from T before starting the construction of $(N_K, \underline{m}_0)$. Such a transition t is usually called dead in $\underline{m}_0$ (cf. Def. 6.1).

Moreover, even if t is not dead in $\underline{m}_0$ for N it may be dead for $\underline{m}_0$ in N_K. Since this property depends on K, we call such transitions K-dead for $\underline{m}_0$. K-dead transitions can be computed effectively from N and therefore removed before the construction of N_K.

Definition 4.4. Let $N = (P, T, F, B)$ be a P/T-net and K be a right-closed subset of $\mathbb{N}^{|P|}$. A transition $t \in T$ is said to be *K-dead* for $\underline{m} \in \mathbb{N}^{|P|}$, if t is not contained in any firing sequences $w \in F_K(N, \underline{m})$. t is *dead* for $\underline{m}$. if t is K-dead for $\underline{m}$ with $K = \mathbb{N}^{|P|}$.

Definition 4.5. Let $N = (P, T, F, B)$ be a P/T-net, $\underline{m}_0 \in \mathbb{N}^{|P|}$ a marking and $\underline{m}_0 \in K \subseteq \mathbb{N}^{|P|}$, K a right-closed set.

The *K-restricted coverability graph* $G_K(N, \underline{m}_0)$ is defined as $G(N, \underline{m}_0)$ in Definition 3.6 with the following modification:

Replace line (4) by:

(4′) choose $\underline{m} \in$ NODES, $t \in T$ such that $F(t) \leq \underline{m}$, $\operatorname{reg}(m + \Delta(t)) \cap K \neq \emptyset$

and the pair (m, t) has not been considered before;

Lemma 4.6. *Let $N = (P, T, F, B)$ be a P/T-net, $K \subseteq \mathbb{N}^{|P|}$ a right-closed set, and $G_K(N, \underline{m}_0)$ a K-restricted coverability graph of N. Then a transition $t \in T$ is not K-dead in $\underline{m}_0$ if and only if $G_K(N, \underline{m}_0)$ containes a path*

$$\underline{m}_0 = \underline{m}'_0 \xrightarrow[t_1]{} \underline{m}'_1 \xrightarrow[t_2]{} \underline{m}'_2 \rightarrow \ldots \xrightarrow[t_n]{} \underline{m}'_n \text{ such that } t_n = t, n \geq 1$$

Proof. If $t \in T$ is not K-dead in $\underline{m}_0$, then there is a firing sequence $w = t_1 \ldots t_n \in F_K(N, \underline{m}_0)$, $\underline{m}_0(t_1\rangle \underline{m}_1(t_2\rangle \ldots (t_n\rangle \underline{m}_n$ with $t_n = t$ and $\underline{m}_i \in K$ for all $1 \leq i \leq n$.

There is a unique path $\underline{m}_0 \xrightarrow{t_1} \underline{m}'_1 \xrightarrow{t_2} \ldots \xrightarrow{t_n} \underline{m}'_n$ in $G_K(N, \underline{m}_0)$ with $(\underline{m}'_i(p) \neq \omega \Rightarrow \underline{m}'_i(p) = \underline{m}_i(p))$. This follows directly from the construction in Definitions 3.6 and 4.5. The additional condition in (4′) of Definition 4.5 is satisfied for all $0 \leq i < n$ since $\underline{m}_i \in K$.

The reverse direction follows from the following stronger claim, which will be proved by induction on $n \in \mathbb{N}$:

Claim: For any constant $c \in \mathbb{N}$ and any path

$$\underline{m}_0 = \underline{m}'_0 \xrightarrow{t_1} \underline{m}'_1 \xrightarrow{t_2} \ldots \xrightarrow{t_{n-1}} \underline{m}'_{n-1} \xrightarrow{t_n} \underline{m}'_n$$

in $G_K(N, \underline{m}_0)$ there are markings $\underline{m}_{n-1}$, $\underline{m}_n \in \mathbb{N}^{|P|}$ and a firing sequence $w t_n \in F_K(N, \underline{m}_0)$ satisfying:

a) $\underline{m}_0(w\rangle \underline{m}_{n-1}(t_n\rangle \underline{m}_n$
b) $\underline{m}_{n-1}(p) = \underline{m}'_{n-1}(p)$ if $\underline{m}'_{n-1}(p) \neq \omega$
c) $\underline{m}_{n-1}(p) \geq c$ if $\underline{m}'_{n-1}(p) = \omega$.

If $n = 1$ then we have the path $\underline{m}_0 = \underline{m}'_0 \xrightarrow{t_1} \underline{m}'_1$ and a), b) and c) are valid by $\underline{m}_0(\lambda\rangle \underline{m}_0(t_1\rangle \underline{m}_1$ for $\underline{m}_1 = \underline{m}_0 + \Delta(\mathrm{t}_1)$. By condition $\operatorname{reg}(\underline{m}_0 + \Delta(t_1)) \cap K \neq \emptyset$ in Definition 4.5 also $\underline{m}_1 \in K$ and $t_1 \in F_K(N, \underline{m}_0)$ is valid.

Now assume that the claim holds for all paths of length n. Let $c' \in \mathbb{N}$ be a constant and

$$\underline{m}_0 = \underline{m}'_0 \xrightarrow{t_1} \ldots \xrightarrow{t_{n-1}} \underline{m}'_{n-1} \xrightarrow{t_n} \underline{m}'_n \xrightarrow{t_{n+1}} \underline{m}'_{n+1}$$

be such a path of length $n+1$.

The following is a standard argumentation on coverability graphs and therefore not developped in full detail here (cf.: [29, 3]). We first consider those ω-components $\underline{m}'_n(p)$ which are different from $\underline{m}'_{n-1}(p)$, i.e. $\underline{m}'_n(p) = \omega \neq \underline{m}'_{n-1}(p)$. By the construction of $G_K(N, \underline{m}_0)$ such components result from existence of predecessor nodes $\underline{m}'_x$ such that $\underline{m}'_x \leq \underline{m}'_n$ but $\underline{m}_x(p) < \underline{m}'_n(p)$.

If v_x is the sequence of transition on this path, then $v_x^{c'} \in F_K(N, \underline{m}_n)$ for any marking $\underline{m}_n$ with $\underline{m}_n(p) = \underline{m}'_n(p)$ if $\underline{m}'_n(p) \neq \omega$ and sufficiently many tokens in $\underline{m}_n(p)$ for $\underline{m}'_n(p) = \omega$. Let be c_0 a number that is sufficiently large for all firings $v_{x_1}^{c'} v_{x_2}^{c'} \dots v_{x_g}^{c'} =: v$ associated to the new ω-components in $\underline{m}_n$. For these components p we therefore have $\Delta(v)(p) \geq c'$.

Next we define the constant c to be used in the induction hypothesis by

$$c := c_0 + \tag{1}$$

$$c' + \tag{2}$$

$$\max_{p \in P} \{F(p, t_n)\} + \tag{3}$$

$$\max_{p \in P} \{F(p, t_{n+1})\} + \tag{4}$$

$$\max_{\substack{p \in P \\ \underline{m} \in \operatorname{res}(K)}} \{\underline{m}(p)\} \tag{5}$$

Intuitively, (1) and (3) serve to make v firable in $\underline{m}_n$, (2) serves to satisfy c), (4) serves to make t_{n+1} firable and (5) is used to show the final marking in K.

More formally, by induction hypothesis, there is a firing sequence $w t_n \in F_K(N, \underline{m}_0)$, satisfying a), b) and c) in the claim.

We consider $\hat{w} := w t_n v \in F_K(N, \underline{m}_0)$ and $\underline{m}_0 (\hat{w}\rangle \hat{\underline{m}}_n$ for some $\hat{\underline{m}}_n \in K$.

a') $\underline{m}_0 (\hat{w}\rangle \hat{\underline{m}}_n (t_{n+1}\rangle \hat{\underline{m}}_{n+1}$, where $\hat{\underline{m}}_{n+1} = \hat{\underline{m}}_n + \Delta(t_{n+1})$ follows from the definition of $G_K(N, \underline{m}_0)$ together with (4).

b') $\hat{\underline{m}}_n(p) = \underline{m}'_n(p)$ if $\underline{m}'_n(p) \neq \omega$ holds by the construction of $G_K(N, \underline{m}_0)$

c') $\hat{\underline{m}}_n(p) \geq c'$ if $\underline{m}'_n(p) = \omega = \underline{m}'_{n-1}(p)$ follows from

$$\begin{aligned}
\hat{\underline{m}}_n(p) &= \hat{\underline{m}}_{n-1}(p) + B(p, t_n) - F(p, t_n) \\
&\geq \hat{\underline{m}}_{n-1}(p) - F(p, t_n) \\
&\geq c - F(p, t_n) \\
&\geq c - \max_{p \in P} \{F(p, t_n)\} \geq c'
\end{aligned}$$

$\hat{\underline{m}}_n(p) \geq c'$ if $\underline{m}'_n(p) = \omega \neq \underline{m}'_{n-1}(p)$ follows from (1) and $\Delta(v)(p) \geq c'$ as discussed above.

Finally it remains to prove: $\hat{w} t_{n+1} \in F_K(N, \underline{m}_0)$, or equivalently $\hat{\underline{m}}_{n+1} \in K$. By the construction of $G_K(N, \underline{m}_0)$ in line (5') we have $\operatorname{reg}(\underline{m}'_n + \Delta(t_{n+1})) \cap K \neq \emptyset$, i.e. $\underline{m}'_n + \Delta(t_{n+1}) \geq \underline{m}_{\text{res}}$ for some $\underline{m}_{\text{res}} \in \operatorname{res}(K)$.

If $\underline{m}'_n(p) \neq \omega$ then by b'):

$$\begin{aligned}
\hat{\underline{m}}_{n+1}(p) &= \hat{\underline{m}}_n(p) + \Delta(t_{n+1})(p) \\
&= \underline{m}'_n(p) + \Delta(t_{n+1})(p) \geq \underline{m}_{\text{res}}(p)
\end{aligned}$$

If $\underline{m}'_n(p) = \omega$ then by c') and (5):

$$\begin{aligned}\underline{\hat{m}}_{n+1}(p) &= \underline{\hat{m}}_n(p) + \Delta(t_{n+1})(p)\\ &\geqq c - F(p, t_n)\\ &\geq \max_{\underline{m}\in \mathrm{res}(K)} \{\underline{m}(p)\} \geqq \underline{m}_{\mathrm{res}}(p).\end{aligned}$$

We conclude: $\underline{\hat{m}}_{n+1} \geqq \underline{m}_{\mathrm{res}}$ and $\underline{\hat{m}}_{n+1} \in K$.

Using the preceding result and remarks we now can modify Construction 4.1 of the K-restriction $(N_K, \underline{m}_0)$ of $(N, \underline{m}_0)$ as follows.

Construction 4.7.

Step 1: Take the notation as in Construction 4.1. Replace T in $(N, \underline{m}_0)$ by $\tilde{T}: = \{t \in T \mid t$ is not K-dead in $\underline{m}_0\}$
Step 2: Define $(N_K, \underline{m}_0)$ or $(N_K, n, \underline{m}_0)$ following Construction 4.1. Using the modified P/T-net $(N, \underline{m}_0)$.
Step 3: Delete all transitions t' from N_K such that $F'(t'') \leqq F(t')$ for some other transition t'' with $h(t'') = h(t')$.
Step 4: Delete all transitions $t_{\underline{m}}$ that are still dead in the initial marking $\underline{m}_0$ of the resulting P/T-net.

The following example shows that step 4 is some times needed.

Example 4.8. Consider the P/T-net $(N, \underline{m}_0)$ of Example 4.3.

$(N_K, \underline{m}_0)$ of Figure 4.1 (b) is also the result of step 1 to 3 in the preceding Construction 4.7.; a and b are not K-dead in $\underline{m}_0$, but $a_{\underline{m}_1}$ and $b_{\underline{m}_2}$ are dead in $\underline{m}_0$, and can be deleted. Figure 4.2 shows $(N_K, \underline{m}_0)$ after having applied Construction 4.7 completely.

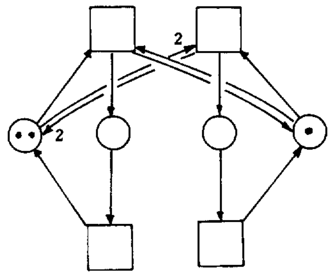

Fig. 4.2. P/T-net $(N_K, \underline{m}_0)$

Another situation that might occur and which is also not easy to detect in general, is the following:

After having constructed a P/T-net $(N_K, \underline{m}_0)$ using Construction 4.7, we know that each transition of $(N_K, \underline{m}_0)$ will be enabled in some reachable marking in $R(N_K, \underline{m}_0)$. But again one can still omit further transitions in N_K without affecting it's behaviour as described in Theorem 4.2.

Let us again give an example that illustrates such a situation.

Example 4.9. Consider the P/T-net $(N, \underline{m}_0)$ from Figure 4.3 where $\underline{m}_0 = (1, 2)$ and the coverability graph $G(N, \underline{m}_0)$ is drawn too.

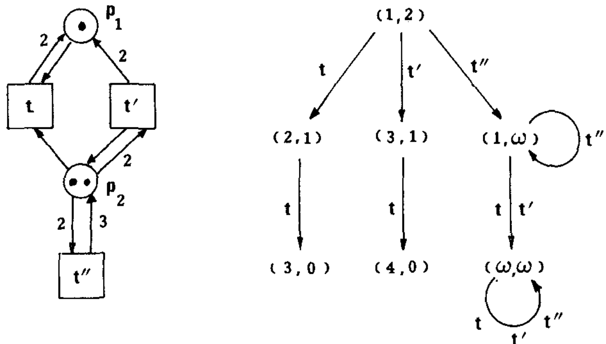

Fig. 4.3.

If the right-closed set $K \subseteq \mathbb{N}^2$ is defined by $\text{res}(K) := \{(0, 2), (3, 0)\}$ then we get the P/T-net $(N_K, \underline{m}_0)$ depicted in Fig. 4.4, where a coverability graph $G(N_K, \underline{m}_0)$ is drawn, too.

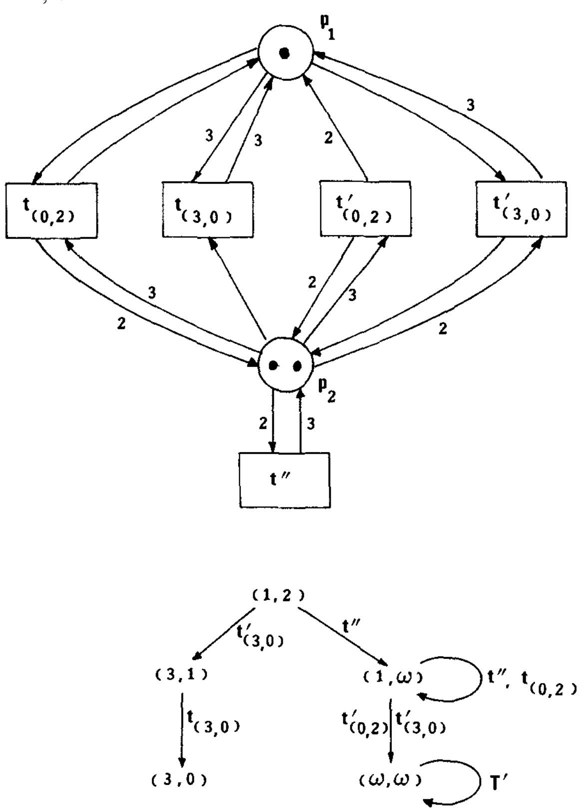

Fig. 4.4. $(N_K, \underline{m}_0)$: Where $T' := \{t'', t_{(0,2)}, t_{(3,0)}, t'_{(0,2)}, t'_{(3,0)}\}$

Now one can show that for all $\underline{m} \in R(N_K, \underline{m}_0)$ the invariant relation $\underline{m}(p_1) \geqq 1$ holds. Hence, whenever $t'_{(0,2)}$ is enabled in $(N_K, \underline{m}_0)$ so is $t'_{(3,0)}$, and thus $t'_{(0,2)}$ can be deleted from T'.

Moreover, we can simplify N_K even more by redefining F' and B' for $t'_{(3,0)}$ as follows:

$$F'(t'_{(3,0)}) := F(t')$$

$$B'(t'_{(3,0)}) := B(t')$$

Hence, we finally get $N_K = (P, T', F', B')$ where $T' := \{t'', t'_{(3,0)}, t_{(0,2)}, t_{(3,0)}\}$ and F' and B' are defined for t'', $t_{(0,2)}$ and $t_{(3,0)}$ as before.

This last example shows that other methods which provide information about the reachability set $R(N, \underline{m}_0)$ might be useful in order to simplity the K-restricted net $(N_K, \underline{m}_0)$ even further.

In many cases one can use place invariants and place invariant inequalities for that purpose. Let us first recall the definition of place invariants.

Definition 4.10. A (linear) *place-invariant for a P/T-net* $N = (P, T, F, B)$ is a vector $\underline{x} \in \mathbb{Z}^{|P|}$ such that $\underline{x}^T \cdot \underline{m} = \underline{x}^T \cdot \underline{m}'$ for all $\underline{m} \in \mathbb{N}^{|P|}$ and all $\underline{m}' \in R(N, \underline{m})$.

This is equivalent to saying $\underline{x}^T \cdot \Delta = \underline{0}^T$, (see [2]) ($\underline{x}^T$ is the transpose of $\underline{x}$).

A (linear) *place-invariant for a P/T-net* $(N, \underline{m}_0)$ *with initial marking* $\underline{m}_0$ is a vector $\underline{x} \in \mathbb{Z}^{|P|}$ such that $\underline{x}^T \cdot \underline{m}_0 = \underline{x}^T \cdot \underline{m}$ for all $\underline{m} \in (N, \underline{m}_0)$.

Obviously, each place-invariant for N is also a place-invariant for $(N, \underline{m}_0)$ but the converse need not be true. If $\underline{x}$ is place-invariant for $(N, \underline{m}_0)$ then $\underline{x}^T \cdot \underline{m}_0$ is a constant $c \in \mathbb{Z}$ so that one obtains an invariant equation of the form

$$\sum_{p \in P} \underline{x}(p) \cdot \underline{m}(p) = c$$

which is valid for all reachable markings $\underline{m} \in R(N, \underline{m}_0)$.

Sometimes it is useful not to consider invariant *equations* of the form described above but to consider invariant *relations* of the form defined below. For example the invariant relation $\underline{m}(p_1) \geqq 1$ was used in Example 4.9 to simplify the K-restricted P/T-net $(N_K, \underline{m}_0)$.

Definition 4.11. A (linear) *place invariant relation for a P/T-net* $N = (P, T, F, B)$ is a vector $\underline{x} \in \mathbb{Z}^{|P|}$, such that

$$\forall \underline{m} \in \mathbb{N}^{|P|} \; \forall \underline{m}' \in R(N, \underline{m}): (\underline{x}^T \cdot \underline{m}' \geqq \underline{x}^T \cdot \underline{m}).$$

A (linear) place invariant relation for a P/T-net $(N, \underline{m}_0)$ with initial marking $\underline{m}_0$ is an $\underline{x} \in Z^{|P|}$, such that $\underline{x}^T \cdot \underline{m}' \geqq \underline{x}^T \cdot \underline{m}_0$ for all $\underline{m}' \in R(N, \underline{m}_0)$.

Obviously, each place-invariant relation for N is also a place-invariant relation for $(N, \underline{m}_0)$ but not conversely.

Lemma 4.7. *Let* N_K (resp $(N_K, \underline{m}_0)$) *be the K-restriction of* N (resp. $(N, \underline{m}_0)$) *using the Construction 4.1 or its modified version 4.6.*

Then every place-invariant (resp. *place-invariant relation*) *for* N (resp. $(N, \underline{m}_0)$) *is also a place-invariant* (resp. *place-invariant relation*) *for* N_K (resp. $(N_K, \underline{m}_0)$).

This Lemma can be applied in Example 4.3:

Invariant equations i_1 and i_2 for $(N, \underline{m}_0)$ also hold for $(N_K, \underline{m}_0)$. Hence transitions $\underline{a}_{m_1}$ and $b_{\underline{m}_2}$ are dead in $\underline{m}_0$ and can be omitted.

5. Maximal Subbehaviour and Liveness of P/T-Nets

In this section we consider K-restricted P/T-nets $(N_K, \underline{m}_0)$ for some particular right-closed sets K. Their behaviour is characterized as the maximal subbehaviour of the original P/T-net $(N, \underline{m}_0)$ with respect to well-defined properties.

The behaviour of a system can be defined as the set all possible sequences of actions. For the definition of "maximal subbehaviour" we therefore use a formalism that is independent of the representation of states: a transition system. Since there is no partial ordering or operation defined on the state space, we cannot use the notation of closed nets etc. The connection to the previous sections will then be established by interpreting marking graphs of P/T-nets as transition systems. Therefore it is sufficient here to consider only initially connected transition systems with finite sets of transitions.

Definition 5.1. A *transition system* $TS = (S, T, \rightarrow, s_0)$ is defined by a set S of *states*, a set T of *transitions*, a *transitional relation* $\rightarrow \subseteq S \times T \times S$, and an *initial state* $s_0 \in S$.

We write $s \xrightarrow{t} s'$ for $(s, t, s') \in \rightarrow$ and extend this notion to words $w \in T^*$ by $s \xrightarrow{\lambda} s$ for all $s \in S$, $s \xrightarrow{wt} s'$ iff $\exists s'' \in S$: $s \xrightarrow{w} s'' \wedge s'' \xrightarrow{t} s'$ for all $s, s' \in S$, $w \in T^*$, $t \in T$.

$R(TS, s) := \{s' \mid \exists w \in T^* : s \xrightarrow{w} s'\}$ is the *set of states reachable from* s and $F(TS, s) := \{w \in T^* \mid \exists s' \in S : s \xrightarrow{w} s'\}$ is the *set of transition sequences* from s. $F_\omega(TS, s) := \{w \in T^\omega \mid$ there is an infinite path $s \xrightarrow{w_1} s_1 \xrightarrow{w_2} s_2 \xrightarrow{w_3} \ldots$ and $w = w_1 w_2 w_3 \ldots\}$ is the set of infinite sequences of transitions from s.

$R(TS) := R(TS, s_0)$, $F(TS) := F(TS, s_0)$ and $F_\omega(TS) := F_\omega(TS, s_0)$ are the sets of *reachable states* in TS, *finite* and *infinite transition sequences* in TS, respectively. In this paper we assume for all transition systems TS, that T is finite and $S = R(TS)$.

Definition 5.2. A transition system $TS = (S, T, \rightarrow, s_0)$ is called

a) *notblocked for* $\hat{T} \subseteq T$, iff for every state $s \in R(TS)$ and some $t \in \hat{T}$ there is a $w \in T^*$ such that $wt \in F(TS, s)$.

b) *notdead*, iff for every state $s \in R(TS)$ the set $F(TS, s)$ is not finite

c) $\hat{T}$*-continual* for a subset $\hat{T} \subseteq T$ iff for every state $s \in R(TS)$ there is an infinite string $w \in F_\omega(TS, s)$ with $\hat{T} \subseteq \mathrm{In}(w)$.

d) *live*, iff for every state $s \in R(TS)$ and every $t \in T$ there is a word $w \in T^*$ such that $wt \in F(TS, s)$.

These properties are not independent, as shown by the following simple theorem.

Theorem 5.3. *A transition system $TS=(S, T, \rightarrow, s_0)$ is notblocked for $\hat{T}$ iff it is $\hat{T}$-continual. TS is T-continual iff TS is live.*

Proof. Let be TS notblocked for $\hat{T}=\{t_1, \ldots, t_r\}$ and $s \in R(TS)$. Then there are words $w_1, \ldots, w_r \in T^*$ and states $s_1, \ldots, s_r \in S$ such that

$$s \xrightarrow{w_1 t_1} s_1 \xrightarrow{w_2 t_2} s_2 \longrightarrow \ldots \xrightarrow{w_r t_r} s_r.$$

Repeating this construction we define inductively an infinite transition sequence w with $\hat{T} \subseteq \text{In}(w)$. Hence TS is $\hat{T}$-continual. The other statements of the Theorem are now obvious.

Definition 5.4. Let $TS=(S, T, \rightarrow, s_0)$ be a transition system. A transition system $TS_i=(S_i, T, \rightarrow_i, s_0)$ is called a *subsystem* of TS iff $S_i \subseteq S$ and $\rightarrow_i \subseteq \rightarrow$.

If $\mathscr{C}(TS)=\{TS_i | i \in I\}$ is a set of such subsystems (with the same initial state), then

$$TS(\mathscr{C})=\{S', T, \rightarrow', s_0\}$$

defined by $S'=\bigcup_{i \in I} S_i$ and $\rightarrow'=\bigcup_{i \in I} \rightarrow_i$ is the *union* of $\mathscr{C}(TS)$. It is the smallest subsystem containing all TS_i as a subsystem, and therefore called *the $\mathscr{C}$-maximal subsystem* of TS. (Recognize that in fact $R(TS(\mathscr{C}))=S'$.)

Definition 5.5. Let $TS=(S, T, \rightarrow, s_0)$ be a transition system. Then we define the following classes of subsystems:

a) *notdead*(TS) is the class of notdead subsystems of TS
b) if $\hat{T} \subseteq T$ then $\hat{T}$-*continual*(TS) is the class of all $\hat{T}$-continual subsystems of TS
c) *live*(TS) is the class of all live subsystems of TS.

Theorem 5.6. *For given transition system $TS=(S, T, \rightarrow, s_0)$ and $\hat{T} \subseteq T$ the classes notdead(TS), $\hat{T}$-continual(TS) and live(TS) are closed under arbitrary union. Hence the notdead-maximal subsystem TS(notdead), the $\hat{T}$-continual-maximal subsystem TS($\hat{T}$-continual), and the live-maximal subsystem TS(live are uniquely defined and not dead, $\hat{T}$-continual and live, respectively.*

Proof. Let be $TS_i=(S_i, T, \rightarrow_i, s_0)$ $(i \in I)$ a set of $\hat{T}$-*continual* $\hat{T}$-continual subsystems of $TS=(S, T, \rightarrow_i s_0)$ and $TS'=(S', T, \rightarrow', s_0)$ the union, i.e. $S'=\bigcup_{i \in I} S_i$ and $\rightarrow'=\bigcup_{i \in I} \rightarrow_i$. For an arbitrary state $s \in S'$ we have to show that there is some $w \in F_\omega(TS, s)$ with $\hat{T} \subseteq In(w)$. But since $s \in S_j$ for some $j \in I$ such an infinite word $w \in F_\omega(TS_i, s)$ exists in TS_i. By the definition of the union of subsystems $w \in F_\omega(TS, s)$ also holds.

The case of the class notdead(TS) is similar but even simpler. For the class *live* (TS) the Theorem follows from the first part the proof as special case $\hat{T} = T$, as stated in Theorem 5.3.

Definition 5.7. Let $(N, \underline{m}_0)$ be a P/T-net $N=(P, T, F, B)$ with initial marking $\underline{m}_0$. To $(N, \underline{m}_0)$ we associate the transition system $TS(N, \underline{m}_0) := (R(N, \underline{m}_0), T, \rightarrow \underline{m}_0)$ where $(\underline{m}_1, t, \underline{m}_2) \in \rightarrow$ iff $\underline{m}_1, \underline{m}_2 \in R(N, \underline{m}_0)$ and $\underline{m}_1 (t\rangle\, \underline{m}_2$.

If $(N, h, \underline{m}_0)$ is a net with labelling homomorphism h: $T^* \rightarrow X^*$ then in $TS(N, \underline{m}_0)$ we replace T by X and t by $h(t)$ in the definition of $\rightarrow$ and write $TS(N, h, \underline{m}_0)$. If $(N, \underline{m}_0)$ is a P/T-net, then we say that a net $(N', \underline{m}_0)$ (resp. $(N', h, \underline{m}_0)$) has the $\mathscr{C}$-maximal subbehaviour of $(N, \underline{m}_0)$ (resp. of $(N, h, \underline{m}_0)$) iff the transition system $TS(N', \underline{m}_0)$ (resp. $TS(N', h, \underline{m}_0)$) is the $\mathscr{C}$-maximal subsystem of $TS(N, \underline{m}_0)$ (resp. of $TS(N, h, \underline{m}_0)$) with $\mathscr{C} \in \{$notdead, $\hat{T}$-continual, live$\}$.

We are now ready to formulate as a Theorem, that for a P/T-net $(N, \underline{m}_0)$ a P/T-net $(N', h, \underline{m}_0)$ with notdead-maximal, $\hat{T}$-continual-maximal or live-maximal subbehaviour can be effectively constructed.

Let us first recall the standard liveness definition for P/T-nets.

Definition 5.8. A P/T-net $(N, \underline{m}_0)$ or $(N, h, \underline{m}_0)$ with $N=(P, T, F, B)$ is *live*, if

$$\forall t \in T\ \forall \underline{m} \in R(N, \underline{m}_0)\ \exists \underline{m}' \in R(N, \underline{m}): \underline{m}'(t\rangle.$$

Theorem 5.9. *For every P/T-net $(N, \underline{m}_0)$ a P/T-net $(N', h, \underline{m}_0)$ can be effectively constructed such that anyone of the following properties holds:*

a) *$(N', h, \underline{m}_0)$ has the notdead-maximal subbehaviour of $(N, \underline{m}_0)$*
b) *$(N', h, \underline{m}_0)$ has the $\hat{T}$-continual-maximal subbehaviour of $(N, \underline{m}_0)$*
c) *$(N', h, \underline{m}_0)$ has the live-maximal subbehaviour of $(N, \underline{m}_0)$*

Proof. In all three cases we define $(N', h, \underline{m}_0)$ as the K-restriction $(N_K, h, \underline{m}_0)$ for different right-closed sets K. (Construction 4.1 or 4.6).

Then $TS' := TS(N_K, h, \underline{m}_0) = (S', T, \rightarrow', \underline{m}_0)$ is a subsystem of $TS := TS(N, \underline{m}_0) = (S, T, \rightarrow, \underline{m}_0)$. Recall that by Construction 4.1 and 4.6 we have $h: T' \rightarrow T$. Since by Theorem 4.2 we have:

$$S' = R(N_K, \underline{m}_0) = R_K(N, \underline{m}_0) \subseteq R(N, \underline{m}_0)$$

and also

$$\left.\begin{array}{l} (\underline{m}_1 \xrightarrow{t}{}' \underline{m}_2 \text{ in } TS') \text{ iff} \\ (\underline{m}_1(t'\rangle\, \underline{m}_2 \text{ in } N_K \text{ and } h(t')=t) \text{ iff} \\ (\underline{m}_1(t\rangle\, \underline{m}_2 \text{ in } N \text{ and } \underline{m}_1, \underline{m}_2 \in K) \text{ iff} \\ (\underline{m}_1 \xrightarrow{t} m_2 \text{ and } \underline{m}_1, \underline{m}_2 \in K) \end{array}\right\} (*)$$

We now prove for $K \in \{$NOTDEAD, CONTINUAL$(\hat{T})\}$ (Def. 3.10) the following statement $(**)$:

$$\left.\begin{array}{l} \text{For every } \underline{m} \in R_K(N, \underline{m}_0) \text{ and } w \in T^{\omega} \\ \text{and } (\hat{T} \subseteq \text{In}(w) \text{ if } K = \text{CONTINUAL}(\hat{T})) \text{ we have:} \\ w \in F_{\omega}(TS, \underline{m}) \text{ implies } w \in F_{\omega}(TS', \underline{m}) \end{array}\right\} (**)$$

Suppose the conditions in $(**)$ and $w \in F_{\omega}(TS', \underline{m})$.

Then with $w = w_1 w_2 \dots, (w_1 \in T^+)$, we have in N

$$\underline{m}(w_1\rangle \underline{m}_1(w_2\rangle \underline{m}_2(w_3\rangle \dots.$$

It follows $\underline{m}_i \in K$ for all $i \in \mathbb{N}$.

Indeed, $w_{i+1} w_{i+2} \dots \in F_\omega(TS, \underline{m}_i)$, hence $F(TS, \underline{m}_i)$ is infinite and $\underline{m}_i \in$ NOTDEAD. If $K = \text{CONTINUAL}(\hat{T})$ then $\hat{T} \subseteq \text{In}(w)$ implies $\hat{T} \subseteq \text{In}(w_{i+1} w_{i+2} \dots)$, hence $\underline{m}_i \in \text{CONTINUAL}(\hat{T})$ for all $i \in \mathbb{N}$. From (*) we conclude $w \in F_\omega(TS' \, \underline{m})$.

Now we consider the cases a), b), and c) of the Theorem.

a) Take $K := \text{NOTDEAD}$ as right-closed set satisfying RES. Then TS' is notdead: if $\underline{m} \in R(TS')$ then $\underline{m} \in S' = R(N', \underline{m}_0) = R_K(N, \underline{m}_0) \subseteq K := \text{NOTDEAD}$. By definition of K the set $F(N, \underline{m})$ is not finite and therefore an infinite sequence $w \in T^\omega$ can fire in $\underline{m}$, i.e. $w \in F_\omega(TS, \underline{m})$. Then by (**) also $w \in F_\omega(TS', \underline{m})$ and $F(TS', \underline{m})$ is infinite too. It remains to show that TS' is the notdead-maximal subsystem of TS, i.e. every notdead subsystem

$$TS_i = (S_i, T, \rightarrow, \underline{m}_0)$$

of TS is also a subsystem of $TS' = (S', T, \rightarrow', m_0)$. Indeed $\underline{m} \in S_i$ implies $\underline{m} \in$ NOTDEAD, and as before $\underline{m} \in S'$, and also $\rightarrow_i \subseteq \rightarrow'$.

b) Take $K := \text{CONTINUAL}(\hat{T})$ as right-closed set satisfying RES. Then TS' is $\hat{T}$-continual: if $\underline{m} \in R(TS')$ then $\underline{m} \in S' = R(N', \underline{m}_0) = R_K(N, \underline{m}_0) \subseteq K = \text{CONTINUAL}(\hat{T})$. By definition of K the set $F_\omega(N, \underline{m})$ contains $w \in T^\omega$ with $\hat{T} \subseteq \text{In}(w)$. Then by (**) also $w \in F_\omega(TS', \underline{m})$ and $\underline{m}$ is $\hat{T}$-continual also in TS'. It remains to show that TS' is the $\hat{T}$-continual-maximal subsystem of TS, i.e. every $\hat{T}$-continual subsystem

$$TS_i = (S_i, T, \rightarrow, \underline{m}_0)$$

of TS is also a subsystem of $TS' = (S', T, \rightarrow', m_0)$. Indeed $\underline{m} \in S'$ implies $\underline{m} \in \text{CONTINUAL}(\hat{T})$, and as before $\underline{m} \in S'$, and also $\rightarrow_i \subseteq \rightarrow'$.

c) Take $K = \text{CONTINUAL}(T)$ as right closed set satisfying RES. TS' is T-continual by b) and live by Theorem 5.3. Every live subsystem TS_i of TS is T-continual, and by a) also a T-continual and live subsystem of TS'. Therefore TS' is the live-maximal subsystem of TS.

Knowing from Theorem 5.3 that a subsystem of TS is $\hat{T}$-continual iff it is notblocked for $\hat{T}$, one might think that in part b) of the preceding proof the choice of $K := \text{NOTBLOCKED}(\hat{T})$ would be equivalent to $K := \text{CONTINUAL}(\hat{T})$.

But this is false. A closer look at the construction of the P/T-net $(N_K, h, \underline{m}_0)$ shows that by inhibiting to fire out of the set NOTBLOCKED($\hat{T}$), also the firing of some $t \in T$ may become impossible! In such a case the transition system is not $\hat{T}$-continual. Let us mention that in case c) of the Theorem $R(N_K, \underline{m}_0)$ and $R(N, \underline{m}_0)$ are equal if $(N, \underline{m}_0)$ is live. By the result of Hack [15], however, it is undecidable whether two nets have the same reachability sets. Therefore our Theorem cannot be used as a decision procedure for liveness of $(N, \underline{m}_0)$.

Example 5.10. The right-closed set $K = \mathrm{CONTINUAL}(T)$ for the P/T-net N in Fig. 5.1 a) has the residue $\mathrm{res}(K) = \{\underline{m}_1, \underline{m}_2, \underline{m}_3, \underline{m}_4\}$ with $\underline{m}_1 := (2,0,0)$, $\underline{m}_2 := (0,1,1)$, $\underline{m}_3 := (1,1,0)$, $\underline{m}_4 := (1,0,1)$.

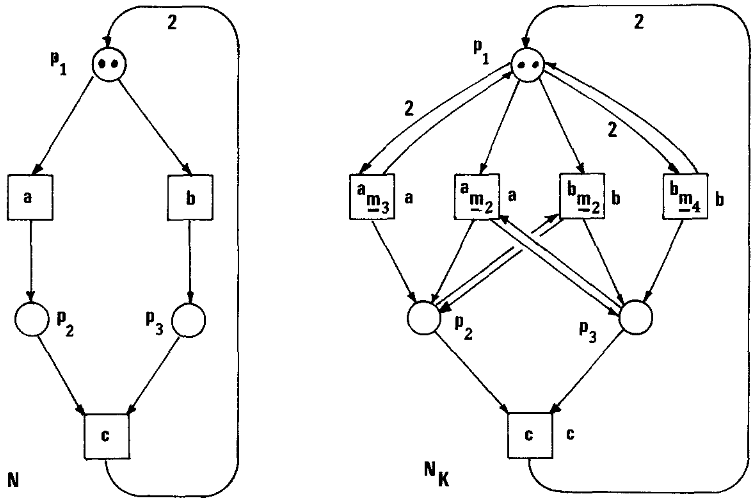

Fig. 5.1.

The K-restriction $(N_K, h, \underline{m}_0)$ of $(N, \underline{m}_0)$ in Fig. 5.1 b) is constructed according to Construction 4.1 and simplifications, obtained by using the invariant equation: $\forall \underline{m} \in R(N, \underline{m}_0)$: $\underline{m}(p_1) + \underline{m}(p_2) + \underline{m}(p_3) = 2$. $(N_K, h, \underline{m}_0)$ has the maximal live subbehaviour of $(N, \underline{m}_0)$.

Theorem 5.9 solves a problem of Nivat and Arnold [22] for the case of P/T-nets. Using our terminology they call a behaviour $F(N, \underline{m}_0)$ *central* if $F(N, \underline{m}_0) \subseteq FG(F_\omega(N, \underline{m}_0))$ where $FG(L)$ is the set of finite prefixes of $L \subseteq X^\omega$. In [22] the problem to realize the maximal central subbehaviour is solved for finite automata and stated as open problem for more powerful devices. Obviously the maximal central subbehaviour is the notdead-maximal subbehaviour in our terminology. Theorem 5.9 also gives a new solution to the older and celebrated banker's problem of Dijkstra [8].

Example 5.11. We demonstrate our approach on the banker's problem, given by Dijkstra in 1965 as an example of a resource sharing problem. For the description of the problem we refer to [1].

Figure 5.2 shows the example in [1] of the banker's problem as a P/T-net. The following invariant equations hold for all $\underline{m} \in R(N, \underline{m}_0)$:

$$i_1: \underline{m}(c) + \underline{m}(l_P) + \underline{m}(l_Q) + \underline{m}(l_R) = 10$$
$$i_2: \underline{m}(l_P) + \underline{m}(c_P) = 8$$
$$i_3: \underline{m}(l_Q) + \underline{m}(c_Q) = 3$$
$$i_4: \underline{m}(l_R) + \underline{m}(c_R) = 9$$

The set of reachable markings $R(N, \underline{m}_0)$ of this net is exactly the set of all markings satisfying the four invariants [16]. Furthermore by these invariants every reachable marking is uniquely determined by the components corresponding to the three places c_P, c_Q and c_R. By the arc from r_P to c_P the initial claim of 8 units in c_P can be restored if the maximal amount of 8 units is loan. The net behaves correctly if *all* customers can perform their transactions. Therefore all reachable markings should be T-continual. In other words, before granting another unit of mony the banker has to verify, that the marking, that would be reached by this step, is still T-continual. For instance in the reachable marking with $(\underline{m}(c_P), \underline{m}(c_Q), \underline{m}(c_R)) = (4, 3, 6)$ an infinite firing sequence can fire (namely $(g_Q g_Q g_Q r_Q)^\omega$), but neither customer P, nor customer Q has a chance to terminate his transactions. Hence the marking m is $\{g_Q, r_Q\}$-continual, but not T-continual.

In [16] general formulas are given, that describe all T-continual markings for such nets. For this example, from 195 reachable markings 60 are not T-continual and contain 24 total deadlocks (c.f. Definition 3.10). The set of 135 reachable and T-continual markings can be described by a residue set R of 10 markings:

$$\begin{aligned} R = \{&(8,2,0), (8,0,2), (2,0,8), (0,2,8),\\ &(7,3,0), (7,0,3), (3,0,7), (0,3,7),\\ &(1,0,9), (0,1,9)\} \end{aligned}$$

This description can be further reduced by observing that R consists of all permutations of $(8,2,0)$, $(7,3,0)$, and $(1,0,9)$ excluding 8 not reachable markings, that do not satisfy the invariants. Hence the banker has to know only these three markings.

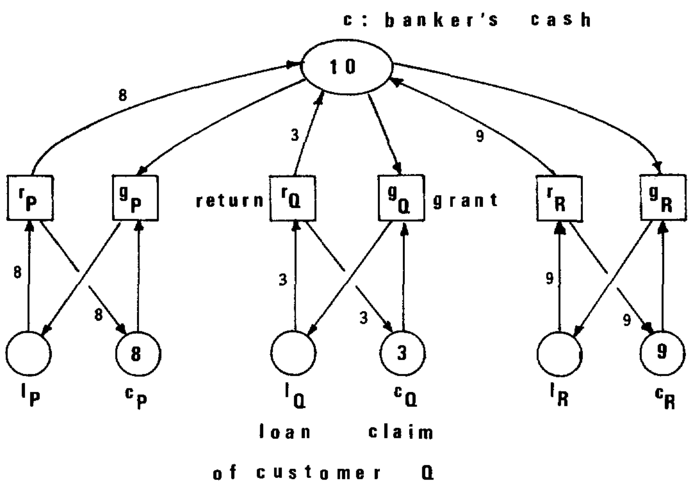

Fig. 5.2.

6. Decidable Properties of Liveness, Promptness and ω-Behaviour of P/T-Nets

In this section we give applications to some other open problems for P/T-nets. In [4] the following definitions for liveness of transitions are proposed according to [18].

Definition 6.1. Let $\underline{m} \in \mathbb{N}^{|P|}$ be a marking of a P/T-net $N=(P,T,F,B)$. Then a transition $t \in T$ is called

a) *dead in* $\underline{m}$ iff $\underline{m}$ is $\{t\}$-blocked.
b) *warm in* $\underline{m}$ iff $\forall n \in \mathbb{N}\ \exists w \in T^*$: $\underline{m}(w\rangle \wedge \Psi(w)(t) > n$.
c) *hot in* $\underline{m}$ iff $\underline{m}$ is $\{t\}$-continual.

Theorem 6.2. *For any P/T-net* $N=(P,T,F,B)$, *any marking* $\underline{m} \in \mathbb{N}^{|P|}$, *and each transition* $t \in T$ *it is decidable, whether t is dead, warm or hot in* $\underline{m}$.

Proof. a) t is dead in a marking $\underline{m}$ of N iff $\underline{m}$ is $\{t\}$-blocked. This is decidable by Corollary 3.12.
b) Add a new place p_{count} to N and let $F(t, p_{\text{count}}) := 1$, in order to count the number of firings of t. Then t is warm in $\underline{m}$ iff p_{count} is unbounded in $\underline{m}$. Unboundedness is decidable. A different proof is contained in [18].
c) t is hot in $\underline{m}$, iff $\underline{m}$ is $\{t\}$-continual. The latter property is again decidable by Corollary 3.12.

We now consider nets that are models of systems communicating with the environment. Actions or transitions that are visible from the exterior are distinguished from internal transitions.

Definitions 6.3. A *signal net* is a P/T-net $N=(P,T,F,B)$ with initial marking $\underline{m}_0$, where $T=T_E \cup T_I$, $T_E \cap T_I = \emptyset$. Transitions in T_E are called *external*, whereas transitions in T_I are *internal*.

An important property of such systems is to react within a finite delay to inputs from the environment. Such systems are called prompt. Compare the similar definitions in [14, 15] using λ-labels for internal transitions.

Definition 6.4. A signal net $N=(P,T,F,B,\underline{m}_0)$ is

a) *strongly prompt*, if $\exists k \in \mathbb{N}\ \forall \underline{m} \in (\underline{m}_0\rangle\ \forall w \in T_I^*$: $\underline{m}(w\rangle \Rightarrow |w| < k$, where $|w|$ is the length of w.
b) *prompt*, if $\forall \underline{m} \in (\underline{m}_0\rangle\ \exists k \in \mathbb{N}\ \forall w \in T_I^*$: $\underline{m}(w\rangle \Rightarrow |w| < k$.

Theorem 6.5. *For a signal net N it is decidable, whether it is strongly prompt and also whether it is prompt.*

Proof. Given a signal net $(N, \underline{m}_0)$ with $N=(P,T,F,B)$, we extend it as shown in Fig. 6.1.

In this construction *all* transitions $t \in T_E$ are connected with p_1 in the given way, and also *all* $t \in T_I$ are connected with p_3 and p_4. The new net is called N', and has the initial marking as N for P and as indicated for the new places.

a) N is not strongly prompt, iff t_3 is warm in the initial marking of N'. To prove this, recall that N is not strongly prompt iff $\forall k \in \mathbb{N}$ $\exists \underline{m} \in (\underline{m}_0\rangle\ \exists w \in T_I^*$: $\underline{m}(w\rangle \wedge |w| > k$. $\underline{m}$ is also reachable in N', if p_1 remains marked and two firings of transitions in T_I are separated by a firing of t_2. Then after firing of t_1 the sequence $w \in T_I^*$ with $|w| > k$ can fire alternating with t_3. Therefore t_3 is warm.

On the other hand, if t_3 is warm in N' a firing sequence w in N' with $\Psi(w)(t) > k$ contains $k-1$ transitions of T_I and no transition of T_E. Hence N is not strongly prompt.

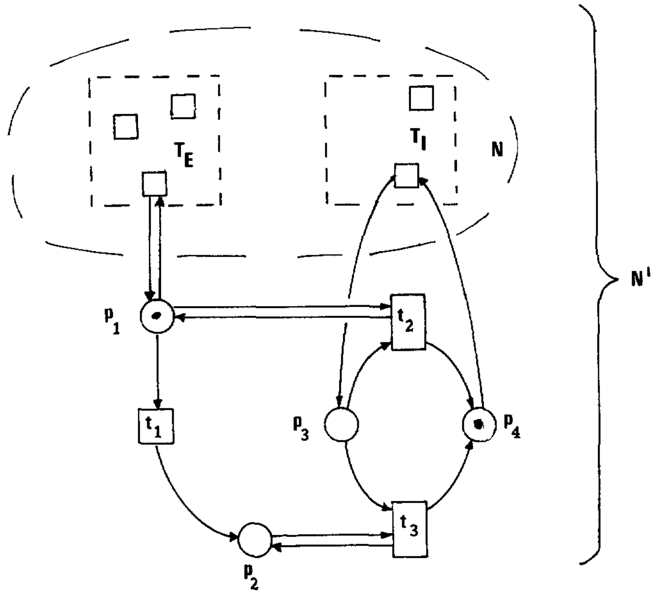

Fig. 6.1.

b) N is not prompt, iff t_3 is hot in the initial marking of N'. N is not prompt iff

$$\exists \underline{m} \in (\underline{m}_0\rangle\ \forall k \in \mathbb{N}\ \exists w \in T_I^*\colon \underline{m}(w\rangle \wedge |w| > k.$$

iff $\exists \underline{m} \in (\underline{m}_0\rangle\colon F(N, \underline{m}) \cap T_I^*$ is infinite

iff $\exists \underline{m} \in (\underline{m}_0\rangle\colon F_\omega(N, \underline{m}) \cap T_I^* \neq \emptyset$

iff t_3 is hot in the initial marking of N'.

By Theorem 6.2 it is decidable, whether t_3 is warm or hot. The first part of the theorem is from [23], who also conjectured the second part, not knowing our results.

Infinite sequences of transition firings have been used already in this paper. The set (or language) of all infinite firing sequences is called the infinite behaviour or ω-behaviour of a net and is systematically studied in [28].

Definition 6.6. For P/T-nets $(N, h, \underline{m}_0)$ with initial marking $\underline{m}_0$ we now also consider not λ-free labelling homomorphisms $h\colon T^* \to X^*$. h is extended to $h\colon T^\infty \to X^\infty$ where $T^\infty := T^* \cup T^\omega$, $X^\infty := X^* \cup X^\omega$ by $h(w)(i) := h(w(i))$ for all $w \in T^\omega$, $i \in \mathbb{N}$ (we assume $h(w) \in X^*$ for $w \in T^\omega$ iff $\exists i\ \forall j \geq i\colon h(w(j)) = \lambda$).

$L_\omega(N, h, \underline{m}_0) = \{h(w) \in X^\omega \mid w \in F_\omega(N, \underline{m}_0)\}$ is the ω-behaviour of $(N, h, \underline{m}_0)$. We consider the following classes of ω-behaviours

$$\mathscr{F}_\omega := \{F_\omega(N, \underline{m}_0) \mid N \text{ is } P/T\text{-net with initial marking } \underline{m}_0\}$$

$$\mathscr{L}_\omega := \left\{\begin{array}{c} L_\omega(N, h, \underline{m}_0) \mid N \text{ is } P/T\text{-net with initial marking } \underline{m}_0 \\ \text{and } \lambda\text{-free labelling homomorphism } h \end{array}\right\}$$

$$\mathscr{L}_{\lambda\omega} := \left\{\begin{array}{c} L_\omega(N, h, \underline{m}_0) \mid N \text{ is } P/T\text{-net with initial marking } \underline{m}_0 \\ \text{and labelling homomorphism } h \end{array}\right\}$$

Theorem 6.7. *The emptiness problem for ω-behaviours in* $\mathscr{F}_\omega$, $\mathscr{L}_{\lambda\omega}$ *and* $\mathscr{L}_\omega$ *is decidable.*

Proof. It is sufficient to consider $\mathscr{L}_{\lambda\omega}$. For any P/T-net $(N, h, \underline{m}_0)$ we have:

$$L_\omega(N, h, \underline{m}_0) \neq \emptyset \text{ iff } \exists t \in T\colon h(t) \neq \lambda \quad \text{and} \quad \underline{m}_0 \text{ is } \{t\}\text{-continual.}$$

In connection with this theorem, one may ask for the decidability of the membership problem: "$w \in F_\omega$?" for given $w \in T^\omega$ and N, for instance. However, already the formulation of the problem is difficult: what is the finite representation of $w \in T^\omega$?

In general this representation can be given by a Turing machine computing all prefixes of w. Then the problem is certainly undecidable. More interesting cases are those, where w is given by an ω-regular expression like $w = ab(aab)^\omega$. Some of these ω-words are ω-behaviours of labelled nets.

For such cases the "membership problem" in the following form is decidable:

Theorem 6.8. *For given λ-free labelled P/T-nets* $(N, h, \underline{m}_0)$ *and* $(N', h', \underline{m}_0)$ *it is decidable whether there is some* $w \in L_\omega(N, h, \underline{m}_0)$ *such that* $w \in L_\omega(N', h', \underline{m}_0)$.

Proof. By well-known methods (e.g. [14]) a net $(\hat{N}, \hat{h}, \underline{m}_0)$ can be constructed such that

$$L(\hat{N}, \hat{h}, \underline{\hat{m}}_0) = L(N, h, \underline{m}_0) \cap L(N', h', \underline{m}'_0)$$

Then $w \in L_\omega(\hat{N}, \hat{h}, \underline{\hat{m}}_0)$ iff $\forall i\colon w[i] \in L(\hat{N}, \hat{h}, \underline{m}_0)$ iff $\forall i\colon w[i] \in L(N, h, \underline{\hat{m}}_0) \wedge w[i] \in L(N', h', \underline{m}'_0)$ iff $w \in L_\omega(N, h, \underline{m}_0) \cap L_\omega(N', h', \underline{m}'_0)$. (The first equivalence is false for not λ-free labelled nets, see [28]).

Hence $L_\omega(N, h, \underline{m}_0) \cap L_\omega(N', h', \underline{m}'_0) \neq \emptyset$ iff $L_\omega(\hat{N}, \hat{h}, \underline{\hat{m}}_0) \neq \emptyset$, which is decidable by Theorem 6.7.

For particular problems, as fairness, for instance, it is interesting to consider some subsets of ω-behaviour in nets. In [28] the notion of *i-definability* of Landweber [20] is studied with respect to sets of markings as definable sets, which specify the subset. The resulting classes have been shown to be different if only bounded places are involved in the definition of defining sets [28].

In [5] it was then proved that the latter classes coincide with the classes obtained by defining sets of transitions.

Definition 6.9. Let $(N, h, \underline{m}_0)$ be a λ-free labelled P/T-net with $N = (P, T, F, B)$ and $\mathscr{E} = \{E_1, \ldots, E_k\}$ a set of sets $E_i \subseteq T$. Then an ω-sequence $w \in T^\omega$ is called

a) *1-firing for* $\mathscr{E}$, if $\exists\ E \in \mathscr{E}\ \exists i \in \mathbb{N}\colon w(i) \in E$.
b) *1′-firing for* $\mathscr{E}$, if $\exists E \in \mathscr{E}\ \forall i \in \mathbb{N}\colon w(i) \in E$.
c) *2-firing for* $\mathscr{E}$, if $\exists E \in \mathscr{E}\colon \mathrm{In}(w) \cap E \neq \emptyset$.
d) *2′-firing for* $\mathscr{E}$, if $\exists E \in \mathscr{E}\colon \mathrm{In}(w) \subseteq E$.
e) *3-firing for* $\mathscr{E}$, if $\exists E \in \mathscr{E}\colon \mathrm{In}(w) = E$.
f) *3′-firing for* $\mathscr{E}$, if $\exists E \in \mathscr{E}\colon \mathrm{In}(w) \supseteq E$.

For $i \in \{1, 1', 2, 2', 3, 3'\}$ we define the *transitional i-behaviour of* $(N, h, \underline{m}_0)$ by $K^i_\omega(N, h, \underline{m}_0, \mathscr{E}) = \{h(v) \in X^\omega \mid v \in F_\omega(N, \underline{m}_0) \wedge v \text{ is } i\text{-firing for } \mathscr{E}\}$ and denote the corresponding classes by K^i_ω.

Theorem 6.10. *The emptiness problem is decidable for all classes* $K^i_\omega (i \in \{1, 1', 2, 2', 3, 3'\})$

Proof. It is interesting to see how the decision problem for the different cases is reducible to $\hat{T}$-continuality. For instance, $K^1_\omega(N, h, \underline{m}_0, \mathscr{E}) \neq \emptyset$ iff for some $E \in \mathscr{E}$ a transition $t \in E$ appears on an arc $m \xrightarrow{t} m'$ of the coverability graph $G(N, \underline{m}_0)$ and $\underline{m}'$ is $\{t\}$-continual for some $t' \in T$.

However, it is easier to reduce all five cases to the case $i = 3$. In fact, from the results in [28], [5] and [7], it follows that

$$K^i_\omega \subseteq K^3_\omega \quad \text{for } i \in \{1, 1', 2, 2', 3'\}$$

Furthermore, it is sufficient to consider $\mathscr{E} = \{E\}$. Hence for given P/T-net $(N, h, \underline{m}_0)$, where $N = (P, T, F, B)$ and $E \subseteq T$ the decision problem is:

"is there some $w \in F_\omega(N, \underline{m}_0)$ such that $\mathrm{In}(w) \supseteq E$?"

This is equivalent to:

"is $\underline{m}_0$ E-continual?",

which is decidable by Corollary 3.12.

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