Petri's Axioms of Concurrency
A Selection of Recent Results

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Abstract

Concurrency theory, as developed by Carl Adam Petri, is an axiomatic theory of binary relations of concurrency (co) and causality (li). This work deals with interactions between axioms and studies properties of concurrency structures, the models of this theory. In contrast to other treatments concurrency theory will be investigated in its general form, which does not require an underlying partial order of causality. Some difficulties are illustrated by counterexamples and possible extensions of the original set of axioms are proposed and analyzed.

Topics: Axiomatic theory of concurrency/causality.

1 Introduction

C. A. Petri suggested in [6], that a comprehensive theory of processes and systems can be developed on the basis of a single binary relation co of concurrency, which can be empirically determined and denotes independence, i.e. absence of causal relationship, in real-world processes.

Concurrency theory (CT) in the sense of Petri investigates properties of the co-relation following from sets of axioms, which can be motivated by seemingly unrelated interpretations, including concurrent processes, measurement scales, and physical space-time. Due to the variety of interpretations certain axioms of CT have also been referred to as axioms of coexistence or axioms of indifference. The first axiomatic treatment of certain aspects of CT was published in [7], where co is conceived as the incomparability relation of a partial order (the axiom of order-based causality) satisfying various further axioms. Partial-order-based CT with special emphasis on its physical significance is also the subject of [9]. An essential feature of CT is that the concurrency relation co uniquely determines a causal net and therefore a causal order (both up to inversion) realizing exactly the concurrency constraints given by co. Furthermore, concurrency structures (CSs) are intended to describe endless processes, where endlessness is ensured by the axiom of local extensibility excluding minimal and maximal elements of the associated causal order.

In [8] and [10] a more general treatment of CT is given, the most interesting point being the declaration of the axiom of order-based causality to be dispensable. Actually this idea has already been present in the early work [7], although it has not been formally investigated. This additional freedom admits in particular cyclic models, which are able to represent repetitive, endless processes in a finite form. Axioms and properties of CSs in this general setting have been studied by the authors in [18] and [5].

The formal investigations of CT the authors are aware of are contained in [3], [4], [2], [15] and [16] and are concerned with partial-order-based structures only. Except for the last these works deal mainly with D-continuity, which has been proposed by Petri as a partial order generalization of Dedekind-continuity in [8]. D-continuity captures the two important aspects of gap-freeness and jump-freeness (see [1]) which constitute the essence of Dedekind continuity.

Another work related to CT is the theory of non-sequential processes published in [1], which pursues a bottom-up approach starting out from general and restricted partial orders and leading to causal orders and causal nets. Finally, properties of systems and their processes (and their underlying causal nets) are related. In contrast to the approach of CT *co* is a derived notion (the relation of incomparability). Although some concepts of CT (in particular K-density and D-continuity) are investigated in this context, central axioms and properties (coherence- and irreducibility-requirements) are not treated and, related to this point, the possibility of characterizing processes by *co* alone is not studied.

A major aim of this work is to present a selection of results obtained by the authors in the course of a formal study of CT during the last years. Most of these results can be found in [18] or [5]. The basis of these works are the publications mentioned above and lectures and seminars given by Petri at the university of Hamburg (e.g. [11], [12]), where concurrency theory was presented as a projection of a larger theory of causality, concurrency, and possibility based on three fundamental binary relations li, co, and al.

We do not try to motivate the suitability of concurrency theory for certain applications (as those mentioned above). Instead we will study axioms/properties of concurrency and their relationships from a formal point of view. Moreover, additional axioms that were not present in Petri's original work will be discussed and formally justified by counterexamples and theorems.

2 Concurrency in Synchronization Graphs

CT is concerned with concurrency and causality but not with alternatives. Therefore only restricted nets without branching states (structural conflicts), i.e. synchronization graphs, can be studied in this context.

Definition 1 [Synchronization graphs and causal nets]
Let N be a net according to Def. 51 and x a net element. x is *branching*, iff $|{\bullet}x| \geq 2$ or $|x{\bullet}| \geq 2$. A *synchronization graph* is a net containing only states that are not branching. A net is *acyclic*, iff its flow relation is acyclic. A *causal net* is an acyclic synchronization graph.

To simplify the presentation in this work we allow markings to contain transitions as well as states. These markings capture the situation where some states are marked and some transitions are just occuring. This straightforward generalization is introduced in the following definition.

Definition 2 [Reachability and case classes]
Let N be a synchronization graph. A *(generalized) marking* is any subset of net

elements. We write $M \to M'$, iff there is a transition t such that $\bullet t \subseteq M \wedge \{t\} \cap M = \varnothing \wedge M' = M - \bullet t \cup \{t\}$ or $\{t\} \subseteq M \wedge t\bullet \cap M' = \varnothing \wedge M' = M - \{t\} \cup t\bullet$. In this case M is said to be a *predecessor* of M' and M' is a *successor* of M. A marking M' is *(forward) reachable* from M, iff $M\ (\to)^*\ M'$. $R := ((\to) \cup (\to)^{-1})^*$ is the relation of *full reachability*. The equivalence classes of R are the *(generalized) case classes* of N.

We restrict our attention to synchronization graphs admitting a safe and proper case class, in order to avoid hidden causality due to contacts and superfluous net elements not contributing to the behavior.

Definition 3 [Safety, security, and properness]
Let N be a net and t a transition. t has a *contact* at M iff $t \in M$ and $(\bullet t \cup t\bullet) \cap M \neq \varnothing$. t has a *transjunction* at M, iff $t\bullet \cap M \neq \varnothing \wedge \bullet t \cap M \neq \varnothing$. Let $\mathcal{C}$ be a case class of N. $\mathcal{C}$ is *safe* (*secure*), iff no transition has a contact (transjunction) at some case of $\mathcal{C}$. $\mathcal{C}$ is *proper*, iff every transition is contained in some case of $\mathcal{C}$ and for every state s there are cases $C, C' \in \mathcal{C}$ with $s \in C$ and $s \notin C'$.

For a causal net concurrency and causality relations can be immediately obtained from its causal order. This is essentially due to the fact that there is no more than one safe and proper case class. For the larger class of synchronization graphs a safe and proper case class is not necessarily unique, which means that, if dynamic properties are of interest, an explicit specification of the case class is required, e.g. by specifying one of its cases.

Definition 4 [*co* and *li* in causal nets]
With a causal net N the partial order $(<) := F^+$ (the *causal order* of N) is associated. Moreover, a *concurrency relation co* and a *causality relation li* are defined by $li := (<) \cup (<)^{-1}$ and $co := X \times X - li - \mathrm{id}_X$.

Definition 5 [*co* and *li* in synchronization graphs]
Let N be a synchronization graph, $\mathcal{C}$ a safe and proper case class, and x, y two different net elements. x and y are said to be *concurrent* in $\mathcal{C}$ ($x\ co\ y$), iff there is some case $C \in \mathcal{C}$ such that $x, y \in C$. x and y are *causally dependent* in $\mathcal{C}$ ($x\ li\ y$), iff x and y are not concurrent in $\mathcal{C}$.

The latter definition is appropriate for synchronization graphs admitting a safe and proper case class. In particular, the proper subclass of (possibly infinite) processes of finite net systems having no transitions with empty preset or empty postset falls into this category. In this case co and li associated with (the causal order of) the causal net are identical to co and li induced by the case class.

As an example consider the synchronization graphs in Fig. 2 and in Fig. 4. The latter one is a causal net, which is extended infinitely to the left and right. For both nets a safe and proper case class is indicated by one of its cases.

Using Def. 5 we obtain the corresponding concurrency relations co, which are shown explicitly in Figs. 1 and 3. li is given by $X \times X - co - \mathrm{id}_X$ and can be easily imagined. Applying Def. 4 to the causal net in Fig. 4 yields exactly the same relations co and li.

From a different perspective synchronization graphs together with their case classes can be conceived as representations of processes. In fact, the processes represented by the two examples are equivalent. The advantage of Fig. 2 is the finiteness of the representation.

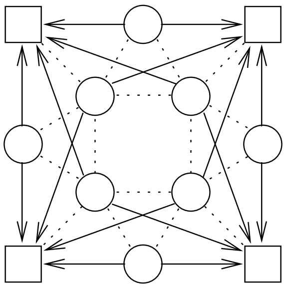

Figure 1: A cyclic model, co and P

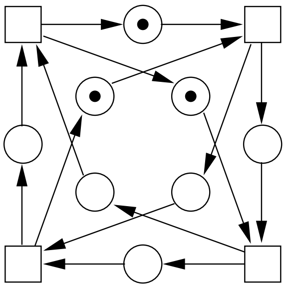

Figure 2: A cyclic model, F

Actually, these two examples (more precisely the triples (X, co, li)) are models of CT, which appear already in Petri's early works. It will turn out that both synchronization graphs and the specified case classes can be reconstructed on the basis of (X, co, li) alone.

CT tries to provide a theory of processes by axiomizing abstract properties of (X, co, li). A major goal is the representability of such a process as a synchronization graph with an appropriate case class.

3 Petri's Axioms of Concurrency

Here most of the concurrency axioms, presented by Petri in the works mentioned above, are defined as (desirable) properties of CSs. A very general notion of concurrency structure is chosen in order to study the strength of different requirements: It consists of a universe X augmented by concurrency and causality relations $co, li \subseteq X \times X$. Although several interpretations of co and li are possible, we will (very briefly) explain the axioms on the basis of the vivid process interpretation, where X denotes a set of states and events, li is interpreted as causal dependence or strict alternation and co as causal independence or concurrency between different elements of X.

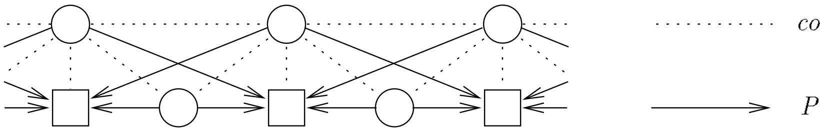

Figure 3: Infinite chain, co and P

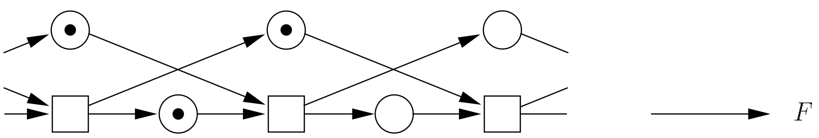

Figure 4: Infinite chain, F

Definition 6 [Concurrency Structures]
Let X be a set and co and li binary relations. (X, co, li) is a *concurrency structure* (CS), iff the following axioms NTR, DIS, COM, and SYM are satisfied.

NTR $|X| \geq 2.$ [Non-triviality]
DIS $co \cap li = li \cap \mathrm{id}_X = co \cap \mathrm{id}_X = \varnothing.$ [Disjointness]
COM $co \cup li \cup \mathrm{id}_X = X \times X.$ [Completeness]
SYM $co^{-1} = co.$ [Symmetry]

Notice that co and li are irreflexive relations in accordance with Petri's latest presentations of the subject in [11] and [12].

For the following it is assumed that (X, co, li) is a CS, that is, NTR, DIS, COM, and SYM are implicitly assumed for all statements. Additional assumptions for a proof are listed explicitly at the beginning of each statement, so that we can check the validity of a proof in any given axiomatic system later on.

A trivial but noteworthy point is, that symmetry of concurrency implies symmetry of causality, $li = li^{-1}$, showing that the direction of causality is not assumed to be relevant at this level.

Proposition 7 **(1)** $li^{-1} = li$; **(2)** $li = X \times X - \underline{co}_X$; **(3)** $co = X \times X - \underline{li}_X$. □

In order to get an intuition for the following definitions and axioms, one might consider the CSs (X, co, li) in Figs. 1 and 3 as examples. Both of them satisfy all the *basic concurrency axioms* which will be introduced now.

LII $\forall x, y \in X : \underline{li}_X[x] = \underline{li}_X[y] \Rightarrow x = y.$ [li-irreducibility]
COI $\forall x, y \in X : \underline{co}_X[x] = \underline{co}_X[y] \Rightarrow x = y.$ [co-irreducibility]

Together li- and co-irreducibility ensure that the elements do not have any identity in addition to their role in the CS. Two different elements are required to be distinguishable in terms of their concurrency *and* causality relations to other elements. Otherwise the elements could be identified and the CS could be reduced. In other words, CT deals only with the irreducible and therefore non-trivial part of behavior.

LIC $li_X^* = X \times X.$ [li-coherence]
COC $co_X^* = X \times X.$ [co-coherence]

li-coherence requires the existence of a weak interaction in form of a finite causality chain between every two elements, otherwise the structure could be partitioned in completely independent components wrt. li_X^*, which could be studied separately. co-coherence is the dual of li-coherence: Between every two elements there should be a chain of concurrency, i.e. a chain of pairwise compatible elements. This requirement is necessary for the existence of a smooth process connecting both elements.

Definition 8 [Lines and cuts]
(1) Lines := $\mathrm{Kens}(\underline{li}_X)$; **(2)** Cuts := $\mathrm{Kens}(\underline{co}_X)$.

Lines are maximal cliques of causality intended to denote maximal sequential subprocesses. Cuts are maximal cliques of concurrency. The intention is, that they represent exactly the global states that may be traversed during the process.

NDE $\forall a, b, c, d \in X :$
$(c\ co\ b \wedge b\ co\ a \wedge a\ co\ d \wedge a\ li\ c \wedge c\ li\ d \wedge d\ li\ b \Rightarrow$
$\exists e \in X : e\ co\ a \wedge e\ co\ b \wedge e\ li\ c \wedge e\ li\ d).$ [N-density]
KDE $\forall C \in \mathrm{Cuts} : \forall L \in \mathrm{Lines} : C \cap L \neq \varnothing.$ [K-density]

With the interpretation of cuts and lines above the property of K-density is quite intuitive: Every maximal sequential subprocess should be visible in every global state. Visibility is witnessed by a common element. N-density is simply a necessary condition (the local aspect) of K-density.

Definition 9 [Proximity, immediate neighborhood, states, and transitions]
(1) $P := \{(x,y) \mid \underline{li}_X[x] \subsetneq \underline{li}_X[y]\}$; **(2)** $im := P \cup P^{-1}$;
(3) $S := \mathrm{dom}\,(P)$; **(4)** $T := \mathrm{ran}\,(P)$.

These definitions may seem strange, as they introduce binary relations of proximity and immediate neighborhood for a CS, which seemingly does not contain any topological or even metric information. That these definitions are indeed reasonable will be supported by formal results presented in subsequent sections. The following intuitive explanation should be sufficient at this stage: The proximity relation between two elements, $x \ P \ y$, can be interpreted as the statement that x is changed by y, as every maximal sequential subprocess passing through x is also passing through y indicating that x and y are tightly connected, i.e. immediate neighbors ($x \ im \ y$). If elements are classified into state-like and transition-like elements, the element y can be seen as a transition (an element of T) denoting the interaction between subprocesses and x (as an immediate neighbor) is a state (an element of S) changed by y. Altogether, im expresses a notion of locality based on concurrency and causality alone.

The P-relations of our examples are shown in Figs. 1 and 3.

NCC $P^2 = \varnothing$. [No changes of changes]
IMC $im^*_X = X \times X$. [im-coherence]

Once the aforementioned interpretations are accepted, these two axioms are easy to understand: No changes of changes simply requires disjointness of states and transitions. im-coherence forces a CS to be connected in terms of the immediate neighborhood relation, that is, between every pair of elements there should be a chain of local interactions.

A further desirable property for processes is the existence of a consistent orientation F directing the symmetric relation of immediate neighborhood im in a way (locally) consistent with concurrency and causality relations. A consistent orientation may be seen as a candidate for a relation indicating the arrow of time.

Definition 10 [Consistent orientation]
A relation $F \subseteq X \times X$ is a *consistent orientation*, iff the following conditions hold: $F \cup F^{-1} = im$ (consistency of changes), $F \odot F \subseteq li$ (li between pre- and postset), $F \odot F^{-1} \subseteq \underline{co}_X$ ($\underline{co}_X$ within preset), $F^{-1} \odot F \subseteq \underline{co}_X$ ($\underline{co}_X$ within postset).
Orient $:= \{F \subseteq X \times X \mid F$ is a consistent orientation $\}$.

The last three conditions express that from the point of view of some element x the immediate past ($F^{-1}[x]$) and the immediate future ($F[x]$) are causally dependent, whereas pairs of elements in the immediate past and pairs of elements in the immediate future are concurrent. Consistent orientations for our examples are given in Figs. 2 and 4, respectively.

COR Orient $\neq \varnothing$. [Consistent orientability]

The global property of consistent orientability, i.e. the existence of at least one consistent orientation, can be weakened to the following local properties of co-transitivity and orientability, which are necessary conditions for consistent orientability. In contrast to global properties, local properties can be verified within the immediate neighborhood $im[x]$ of some element x. The property of local extensibility requires, that with respect to some element in the immediate past of x there is some element in the immediate future and vice versa.

LCT $\quad \forall x \in X : (co|_{im[x]})^2 \subseteq \underline{co}_X|_{im[x]}$. [Local *co*-transitivity]

LOR $\quad \forall x \in X : (li|_{im[x]})^2 \subseteq \underline{co}_X|_{im[x]}$. [Local orientability]

LEX $\quad \forall x \in X : \mathrm{id}_{im[x]} \subseteq (li|_{im[x]})^2$. [Local extensibility]

The properties defined so far except for the global requirement COR will be referred to as *basic concurrency axioms*.

If causality can be derived from a partial order $(<)$ by $li = (<) \cup (<)^{-1}$, this order can be interpreted as a causal order, where $x < y$ means that y causally depends on x.

Definition 11 [Causal order]
A partial order $(<) \subseteq X \times X$ is a *causal order*, iff $(<) \cup (<)^{-1} = li$.
Order $:= \{(<) \subseteq X \times X \mid (<)$ is a causal order$\}$.

CSs admitting such a causal order are said to exhibit order-based causality. CSs in this class are significantly easier to handle, because the associated partial order can be exploited in many proofs.

OBC $\quad$ Order $\neq \varnothing$. [Order-based causality]

Two classes of CSs introduced in [18] are of major interest: *cyclic* CSs, satisfying LFI and *acyclic* CSs satisfying LIN.

LFI $\quad \forall L \in$ Lines : L is finite. [Line finiteness]

LIN $\quad \forall L \in$ Lines : L is infinite. [Line infiniteness]

Both of these classes are not empty as the examples demonstrate.

The CS in Fig. 3 satisfies all the basic concurrency axioms and additionally order-based causality and line infiniteness. The derived proximity relation P is shown in the same figure. The immediate neighborhood $im = P \cup P^{-1}$ can be easily inferred. With the consistent orientation F given in Fig. 4 we can construct a causal order $(<) = F^+$ and an (endless) causal net (S, T, F) realizing exactly those concurrency and causality relations specified by co and li.

The CS in Fig. 1 satisfies all basic concurrency axioms and also line finiteness. It does, however, not exhibit order-based causality. It is a model with the smallest number of elements, as it is proved in [5]. As in the previous example P, S, T, and im can be derived and a consistent orientation F (shown in Fig. 2) can be found. The resulting net (S, T, F) is a synchronization graph and there is a case class (represented e.g. by the marking shown), which realizes exactly those concurrency and causality relations specified by co and li.

In both CSs it is moreover true that the set of cuts is identical to the only safe and proper case class of the synchronization graph (S, T, F). Hence, we can completely recover the net structure and the case class from (X, co, li).

4 Elementary Properties

Most of the following properties are straightforward and given without proof.

Proposition 12 **(1)** $\forall K \in$ Cliques$(\underline{co}_X)$: $\exists C \in$ Cuts : $K \subseteq C$; **(2)** $\forall K \in$ Cliques$(\underline{li}_X)$: $\exists L \in$ Lines : $K \subseteq L$; **(3)** KDE implies NDE. $\square$

Proposition 13 **(1)** $P \subseteq S \times T$; **(2)** $P \cap P^{-1} = \varnothing$; **(3)** $P \cap \mathrm{id}_X = \varnothing$; **(4)** $P \subseteq li$; **(5)** $im \subseteq li$; **(6)** $im = im^{-1}$. □

Proposition 14 **(1)** $P \odot co \subseteq co$; **(2)** $\underline{li}_X \odot P \subseteq \underline{li}_X$. **(3)** $P \subseteq co \odot li$; **(4)** $P \cap li \odot co = \varnothing$; **(5)** $P \subseteq li - li \odot co$; **(6)** $\boxed{\text{LII}}$ $li - li \odot co \subseteq P$; **(7)** $\boxed{\text{LII}}$ $P = li - li \odot co$.

Proof **Part (1)** Let $x\ P\ y\ co\ z$, then $\underline{li}_X[x] \subseteq \underline{li}_X[y]$ and $\neg y\ \underline{li}_X\ z$. Hence $\neg y\ \underline{li}_X\ z$ and $y\ co\ z$. **Part (2)** Let $x\ \underline{li}_X\ y\ P\ z$, then $\underline{li}_X[y] \subseteq \underline{li}_X[z]$ and thus $x\ \underline{li}_X\ z$. **Part (3)** Let $x, y \in X$ and $x\ P\ y$, then $\underline{li}_X[x] \subsetneq \underline{li}_X[y]$. Therefore a $z \in X$ with $y\ \underline{li}_X\ z$ and $\neg x\ \underline{li}_X\ z$ exists. We have $x\ co\ z$. Because of $x\ li\ y$, we conclude $y \neq z$ and with $y\ \underline{li}_X\ z$ we get $y\ li\ z$. Hence $x\ co \odot li\ y$. **Part (4)** Assume the contrary. Then there are $x, y, z \in X$ such that $x\ P\ y$ and $x\ li\ z\ co\ y$. Obviously, $x\ \underline{li}_X\ z$. Now $\underline{li}_X[x] \subseteq \underline{li}_X[y]$ requires $y\ \underline{li}_X\ z$ which leads to $\neg y\ co\ z$, a contradiction. **Part (5)** $P = (P - li \odot co) \cup (P \cap li \odot co) = P - li \odot co \subseteq li - li \odot co$, because of Prop. 13 (4). **Part (6)** Choose arbitrary $x, y \in X$ with $x\ li\ y$ and $\neg x\ li \odot co\ y$. So we have $\forall z \in X : x\ li\ z \Rightarrow y\ \underline{li}_X\ z$. We know $x\ li\ y$. Therefore we get $\forall z \in X : x\ \underline{li}_X\ z \Rightarrow y\ \underline{li}_X\ z$. Hence $\underline{li}_X[x] \subseteq \underline{li}_X[y]$. Because $x \neq y$ is clear from $x\ li\ y$, we get $\underline{li}_X[x] \neq \underline{li}_X[y]$ with the aid of Axiom LII. Altogether $\underline{li}_X[x] \subsetneq \underline{li}_X[y]$ and $x\ P\ y$. **Part (7)** By (5) and (6). □

Proposition 15 **(1)** $\boxed{\text{IMC}}$ $\forall x \in X : im[x] \neq \varnothing$; **(2)** $\boxed{\text{IMC}}$ $\boxed{\text{LEX}}$ $\forall x \in X : |im[x]| \geq 2$; **(3)** $\boxed{\text{NCC}}$ $S \cap T = \emptyset$; **(4)** $\boxed{\text{IMC}}$ $S \cup T = X$.

Proof **Part (1)** By Axiom IMC and Axiom NTR. **Part (2)** By (1) there is $a \in im[x]$. Axiom LEX yields a further element in $im[x]$. **Part (3)** $x \in S \cap T$ violates Axiom NCC. **(5)** $S \cup T \subseteq X$ is clear. $X \subseteq S \cup T$ holds due to (1). □

5 Local Propagation Rules

Under some weak conditions it is possible to show that the binary relations of concurrency and causality propagate through the structure according to certain local rules. One should note that neither consistent orientability nor order-based causality are necessary to derive these results.

For an arbitrary element x, which is concurrent to a transition t, concurrency can be propagated in such a way that x is concurrent to all immediate neighbors $im[t]$ of the transition. A dual propagation rule holds for causality and states.

Proposition 16 $\boxed{\text{NCC}}$ Let $s \in S$, $t \in T$, and $x \in X$.
(1) $x\ co\ t \Rightarrow (\forall y \in im[t] : x\ co\ y)$; **(2)** $x\ \underline{li}_X\ s \Rightarrow (\forall y \in im[s] : x\ \underline{li}_X\ y)$;
(3) For all lines L with $s \in L$: $im[s] \subseteq L$.

Proof **Part (1)** Let $x\ co\ t\ im\ y$, then Prop. 13 (1) and Prop. 15 (3) give us $y\ P\ t$. Now Prop. 14 (1) leads to $y\ co\ x$. **Part (2)** Let $x\ \underline{li}_X\ s\ im\ y$, then Prop. 13 (1) and Prop. 15 (3) give us $s\ P\ y$. Now Prop. 14 (2) leads to $x\ \underline{li}_X\ y$. **Part (3)** Choose any $s \in L$ and $x \in im[s]$. By Prop. 13 (1) and Prop. 15 (3) $s\ P\ x$, hence $\underline{li}_X[s] \subseteq \underline{li}_X[x]$. Because of $s \in L$, we get $L \subseteq \underline{li}_X[s] \subseteq \underline{li}_X[x]$ and only $x \in L$ is possible. □

An idea which can be found in [11] is the following: Using symmetry and the local axioms we show that $\underline{co}_X$ is an equivalence relation on the immediate neighborhood of each element x. It induces a partition of $im[x]$ into exactly two equivalence classes, corresponding to the immediate past and future of x.

Proposition 17 **(1)** $\boxed{\text{LCT}}$ $\forall x \in X : \underline{co}_X$ is an equivalence on $im[x]$; **(2)** $\boxed{\text{IMC}}$ $\boxed{\text{LCT}}$ $\boxed{\text{LEX}}$ $\forall x \in X : |im[x]/\underline{co}_X| \geq 2$; **(3)** $\boxed{\text{LCT}}$ $\boxed{\text{LOR}}$ $\forall x \in X : |im[x]/\underline{co}_X| \leq 2$; **(4)** $\boxed{\text{IMC}}$ $\boxed{\text{LCT}}$ $\boxed{\text{LOR}}$ $\boxed{\text{LEX}}$ $\forall x \in X : |im[x]/\underline{co}_X| = 2$.

Proof **Part (1)** Reflexivity of $\underline{co}_X$ is clear. Symmetry and transitivity within $im[x]$ follow from Axiom SYM and Axiom LCT, respectively. **Part (2)** Use Prop. 15 (1) to find an element $a \in im[x]$. Axiom LEX yields a further element $b \in im\ [x]$ with $a\ li\ b$. **Part (3)** Assume there are more than two equivalence classes. Then there are a, $b, c \in im[x]$ with $a\ li\ b\ li\ c\ li\ a$ violating Axiom LOR. **Part (4)** Combine (2) and (3). $\square$

With the help of the local $\underline{co}_X$-equivalence classes we can formulate two other somewhat complicated but essential propagation rules. We will start with some additional lemmata, whose full power will become apparent later on.

Lemma 18 $\boxed{\text{KDE}}$
$\forall L \in \text{Lines} : \forall x \in L : \forall y \in X : (y\ P\ x \Rightarrow \exists z \in im[x] \cap \underline{co}_X[y] : z \in L)$.
Proof Choose any $L \in \text{Lines}$, $x \in L$, and $y \in X$ with $y\ P\ x$. Let $C \in \text{Cuts}$ be a cut with $x \in C$. Let $K = C - \{x\}$, then $\forall k \in K : x\ co\ k$. Because of $y\ P\ x$ we apply Prop. 14 (1) and get $\forall k \in K : y\ co\ k$. Therefore $K \cup \{y\}$ is a $\underline{co}_X$-Clique, that can be expanded to a cut C' with $K \cup \{y\} \subseteq C'$. We have $L \cap C' \neq \varnothing$ due to Axiom KDE. Choose some $z \in L \cap C'$. Because $\{y, z\} \subseteq C'$ we get $y\ \underline{co}_X\ z$. Assume $\neg z\ im\ x$. Since $y\ im\ x$, clearly $y \neq z$. Hence $y\ co\ z$. Furthermore $y\ im\ x$ leads to $x\ li\ y$. This entails $z \neq x$. $\{z, x\} \subseteq L$ now leads to $z\ li\ x$. We have $\neg \underline{li}_X[z] \subsetneq \underline{li}_X[x]$ due to $\neg z\ P\ x$. As $x\ \underline{li}_X\ y$ and $z\ co\ y$, we get $\underline{li}_X[z] \neq \underline{li}_X[x]$. Combining the two results we get $\neg \underline{li}_X[z] \subseteq \underline{li}_X[x]$. Hence, there is a $u \in X$ with $z\ \underline{li}_X\ u$ and $\neg x\ \underline{li}_X\ u$. We get $x\ co\ u$. As $z\ li\ x$, we know that $z \neq u$, so we can deduce $z\ li\ u$. Let $L' \in \text{Lines}$ be a line such that $\{z, u\} \subseteq L'$. Axiom KDE implies $L' \cap C \neq \varnothing$, therefore we can find a $v \in L' \cap C$. If $v = x$, then $v = x\ co\ u$ in contradiction with $\{u, v\} \subseteq L'$, hence $v \neq x$. But as $v \in C$, this means $v \in K$ or $v \in C'$. Now $\{v, z\} \subseteq C'$ leads to $v\ \underline{co}_X\ z$. Due to $\{v, z\} \subseteq L'$ we get $v\ \underline{li}_X\ z$, hence $v = z$. But now $v = z\ li\ x$ is in contradiction with $\{v, x\} \subseteq C$. Therefore $z\ im\ x$, proving the lemma. $\square$

Lemma 19 $\boxed{\text{KDE}}$
$\forall L \in \text{Lines} : \forall x \in L : \forall y \in im[x] : \exists z \in im[x] \cap \underline{co}_X[y] : z \in L$.
Proof Let $L \in \text{Lines}$, $x \in L$, $y \in X$, and $x\ im\ y$. If $y\ P\ x$, then we can apply Lemma 18. So let us assume $x\ P\ y$. $L \subseteq \underline{li}_X[x]$ due to $x \in L$. We get $L \subseteq \underline{li}_X[y]$ with the help of $x\ P\ y$, hence $y \in L$. The lemma follows with $z = y$. $\square$

In [11] the proof of the following lemma was left as an exercise, which turned out to be quite involved. Note that our proof depends on Lemma 18.

Lemma 20 $\boxed{\text{KDE}}$ $P \subseteq co \odot P$.
Proof Assume $y\ P\ x$. Prop. 14 (3) proves the existence of a $u \in X$ with $y\ co\ u$ and $u\ li\ x$. Let $L \in \text{Lines}$ be a line with $x, u \in L$. Due to Lemma 18 there is a $z \in im[x] \cap \underline{co}_X[y]$ with $z \in L$. We get $z\ \underline{li}_X\ u$ and $z \neq y$ because of $y\ co\ u$. Combining the results we get $z\ co\ y$. Either $x\ P\ z$ or $z\ P\ x$. If $x\ P\ z$, then $z\ \underline{li}_X\ y$ because of $x\ \underline{li}_X\ y$, in contradiction to $z\ co\ y$. Hence $y\ co\ z\ P\ x$. $\square$

With the help of the following theorem it it possible to characterize states and transitions and distinguish them in local terms: The immediate neighborhood of states has exactly two elements partitioned into two singleton equivalence classes, whereas the immediate neighborhood of transitions has at least four elements partitioned into two non-singleton equivalence classes.

Proposition 21 Let $s \in S$ and $t \in T$. **(1)** $\boxed{\text{LCT}}$ $\boxed{\text{NCC}}$ $\forall Q \in (im[s]/\underline{co}_X) : |Q| = 1$; **(2)** $\boxed{\text{LCT}}$ $\boxed{\text{NCC}}$ $\boxed{\text{KDE}}$ $\forall Q \in (im[t]/\underline{co}_X) : |Q| \geq 2$; **(3)** $\boxed{\text{IMC}}$ $\boxed{\text{LCT}}$ $\boxed{\text{LOR}}$ $\boxed{\text{LEX}}$ $\boxed{\text{NCC}}$ $|im[s]| = 2$; **(4)** $\boxed{\text{IMC}}$ $\boxed{\text{LCT}}$ $\boxed{\text{LEX}}$ $\boxed{\text{NCC}}$ $\boxed{\text{KDE}}$ $|im[t]| \geq 4$.
Proof **Part (1)** As an equivalence class is never empty, we have to show $|Q| \leq 1$. So assume there are different $y, z \in Q$ with $y\ co\ z$. By Prop. 13 (1) and Prop. 15 (3) this

implies $s\ P\ y$ and $s\ P\ z$. With $s\ P\ y$ and $y\ co\ z$ Prop. 14 (1) yields $s\ co\ z$ contradicting $s\ P\ z$ (by Prop. 13 (4)). **Part (2)** Choose some $y \in Q$. Prop. 13 (1) and Prop. 15 (3) imply $y\ P\ t$. Lemma 20 yields a z with $y\ co\ z$ and $z\ P\ t$. So $z\ im\ t$, and because $y\ co\ z$, y and z are different elements in the equivalence class Q. **Part (3)** By (1) and Prop. 17 (4). **Part (4)** By (2) and Prop. 17 (2). $\square$

We can now prove the next set of propagation rules: Given two elements x and y, if y is concurrent to all elements in one $\underline{co}_X$-equivalence class Q of $im[x]$, then the element y is also concurrent to x.

Proposition 22 $\boxed{\text{LCT}}$ $\boxed{\text{KDE}}$ Let $x, y \in X$ and $Q \in (im[x]/\underline{co}_X)$. Then **(1)** $x\ \underline{li}_X\ y \Rightarrow (\exists z \in Q : z\ \underline{li}_X\ y)$, **(2)** $(\forall z \in Q : z\ co\ y) \Rightarrow x\ co\ y$.
Proof **Part (1)** Because $x\ \underline{li}_X\ y$, we can find $L \in$ Lines with $x, y \in L$. Choose $w \in Q$, then $w\ im\ x$. By Lemma 19 there is some $z \in im[x] \cap \underline{co}_X\ [w]$ such that $z \in L$. We get $z\ \underline{li}_X\ y$ from $z, y \in L$ and $z \in Q$ from $z\ \underline{co}_X\ w \in Q$. **Part (2)** The contraposition of (1). $\square$

6 The Structure of Lines

By definition lines are just unstructured sets. But already the simple examples suggest that the im-relation induces a distinctive structure for each line.

An element $x \in L$ is called an endpoint of the line $L \in$ Lines, if there is precisely one element $y \in L$ such that $y\ im\ x$. Put differently, it is an endpoint, if $|L \cap im[x]| = 1$. This should be impossible, because Axiom LEX guarantees the local extensibility of the structure. The proof seems trivial, but it makes use of the rather difficult Lemma 19.

Proposition 23 $\boxed{\text{KDE}}$ $\boxed{\text{LEX}}$ $\forall L \in \text{Lines} : \forall x \in L : |im[x] \cap L| \neq 1$.
Proof Assume $L \in$ Lines and $x \in L$ such that $|im[x] \cap L| = 1$. Let $w \in im[x] \cap L$. Because of Axiom LEX there is some $y \in im[x]$ with $y\ li\ w$. By Lemma 19 there exists a $z \in im[x] \cap \underline{co}_X[y]$ such that $z \in L$. Because $\{z, w\} \subseteq im[x] \cap L$, $|im[x] \cap L| = 1$ leads to $z = w$. Now $y\ li\ w = z\ \underline{co}_X\ y$. Contradiction. $\square$

Proposition 24 $\boxed{\text{KDE}}$ $\boxed{\text{IMC}}$ $\forall L \in \text{Lines} : \forall x \in L : |im[x] \cap L| \neq 0$.
Proof Let $L \in$ Lines and $x \in L$. Prop. 15 (1) proves the existence of a $y \in X$ with $x\ im\ y$. By Lemma 19 there exists a $z \in im[x]$ with $z \in L$. $\square$

Lemma 25 $\boxed{\text{LOR}}$ $\forall L \in \text{Lines} : \forall x \in L : |im[x] \cap L| \leq 2$.
Proof Immediate by Axiom LOR. $\square$

Combining the previous results we get the main theorem of this section, stating that every element x of a line L has exactly two immediate neighbors on this line. These elements may be interpreted to correspond to the immediate past and future of x within the sequential subprocess represented by L.

Theorem 26 $\boxed{\text{KDE}}$ $\boxed{\text{IMC}}$ $\boxed{\text{LOR}}$ $\boxed{\text{LEX}}$ $\forall L \in \text{Lines} : \forall x \in L : |im[x] \cap L| = 2$.
Proof Combine Prop. 23, Prop. 24, and Lemma 25. $\square$

Considering Thm. 26 it is tempting to assert that every line can be represented by a two-way infinite im-chain (acyclic case) or by an im-cycle (cyclic case). But as early as in [2] an example of an undesirable line was presented. That line consisted of *two* two-way infinite im-chains that were not connected to each other. Nevertheless the structure satisfied Axiom IMC and all the other basic axioms. Such a line, which we will call a broken line, poses a serious problem

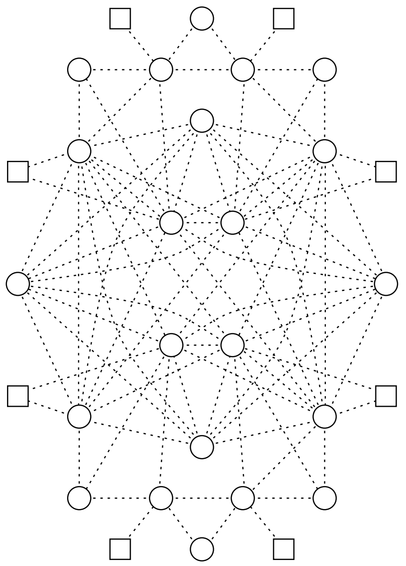

Figure 5: Broken lines, *co*

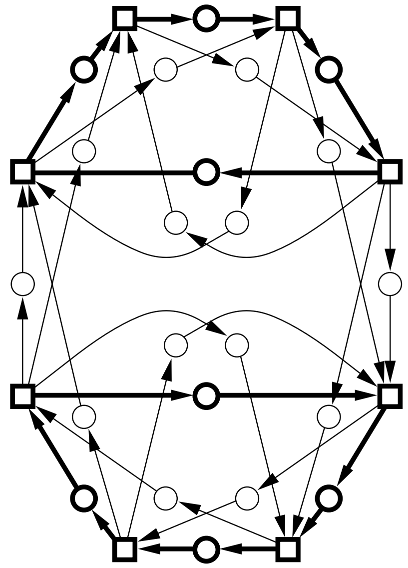

Figure 6: Broken lines, F

when we try to interpret lines as maximal sequential subprocesses. To remove broken lines, which would also violate D-continuity, the aditional axiom

COO $\quad \forall (<) \in \text{Order} : (<)^+ = (<)$ [Combinatorial order]

was proposed in [2]. But Axiom COO does not really catch the spirit of CT, because unlike the other axioms it makes a direct assumption about $(<)$. Moreover, it is only applicable in the order-based case.

But is it possible to have broken lines in cyclic models? It is, as the example in Figs. 5 and 6 shows. It is already quite large, therefore a computer tool was used to prove that the model satisfies all the basic axioms. In Fig. 6 we have highlighted a set of elements and F-arcs. The elements make up a line which falls apart into two unconnected segments at the top and at the bottom of the figure respectively.

There are different conditions that exclude broken lines. For example

LCO $\quad \forall L \in \text{Lines} : (im|_L)^*_L = L \times L,$ [Line coherence]

which was proposed in [18], asserts simply, that two elements of a line can be reached from each other by im-steps within the line.

We can easily rule out lines that are shorter than four elements as the following proposition states. This lower bound is optimal as we can see from Fig. 1 where we can find two lines with exactly four elements.

Proposition 27 $\boxed{\text{IMC}}$ $\boxed{\text{LCT}}$ $\boxed{\text{LEX}}$ $\boxed{\text{NCC}}$ $\boxed{\text{KDE}}$ $\forall L \in \text{Lines} : |L| \geq 4.$ $\square$

The newly presented results (especially the exclusion of endpoints) are powerful tools for reasoning about concurrency structures, however simple they may seem. Axiom LCO was not present in Petri's original works, but it is natural and plausible and allows us to make strong statements about the structure of each individual line. In the order-based case Axiom LCO is similar to the *weak* *discreteness* from [1] which is a necessary precondition for D-continuity. In the cyclic case no replacement axiom is available.

7 Consistent Orientations

As $F \cap F^{-1} = \varnothing$ and $F \cup F^{-1} = im$, a consistent orientation F augments the immediate neighborhood relation im with a direction.

Proposition 28 Let $F \in$ Orient. Then $F \cap F^{-1} = \varnothing$.
Proof Use $F \times F \subseteq li$ (Def. 10) and $li \cap \mathrm{id}_X = \varnothing$. $\square$

The additional requirement that F is locally consistent with concurrency and causality can also be expressed by Petri's four *F-propagation rules* from [11]:

Remark 29 Let $F \in$ Orient.
$x\ F\ y \wedge y\ im\ z \wedge x\ li\ z \Rightarrow y\ F\ z$; $\quad x\ im\ y \wedge y\ F\ z \wedge x\ li\ z \Rightarrow x\ F\ y$;
$x\ F\ y \wedge y\ im\ z \wedge x\ \underline{co}_X\ z \Rightarrow z\ F\ y$; $\quad x\ im\ y \wedge y\ F\ z \wedge x\ \underline{co}_X\ z \Rightarrow y\ F\ x$. $\square$

These rules can be applied to uniquely determine the F-relations between an element x and its immediate neighbors $im[x]$, if one of these F-relations is already known. If some $F \in$ Orient exists, then the local axioms are trivially valid, as the following proposition states.

Lemma 30 Let $F \in$ Orient and $x \in X$. **(1)** $im[x] = {\bullet}x \cup x{\bullet}$;
(2) $\forall y \in {\bullet}x : \forall z \in x{\bullet} : y\ li\ z$; **(3)** $\forall y, z \in {\bullet}x : y\ \underline{co}_X\ z$; **(4)** $\forall y, z \in x{\bullet} : y\ \underline{co}_X\ z$. $\square$

Proposition 31 **(1)** COR implies LCT; **(2)** COR implies LOR.
Proof By Lemma 30. $\square$

It also follows from the definition of a consistent orientation F, that for every element the partitioning of $im[x]$ into equivalence classes wrt. $\underline{co}_X$ coincides with the partitioning of $im[x]$ into pre- and postset wrt. F.

Proposition 32 $\boxed{\text{LEX}}\boxed{\text{IMC}}$ Let $F \in$ Orient, $x \in X$. $(im[x]/\underline{co}_X) = \{{\bullet}x, x{\bullet}\}$.
Proof By Prop. 15 (1) there is a $y \in im[x]$. By Lemma 30 (1) we get $x\ F\ y \vee y\ F\ x$, assume wlog. $y\ F\ x$. Now LEX yields $z \in im[x]$ with $y\ li\ z$, hence $x\ F\ z$ and ${\bullet}x \neq \varnothing \neq x{\bullet}$. Because $F \in$ Orient we have COR and Prop. 31 (1) ensures LCT. Hence $(im[x]/\underline{co}_X)$ is well-defined. Now we proceed using Lemma 30 and Prop. 17 (1). $\square$

With Axiom COR and Axiom OBC we introduced two axioms that supply us with forms of directed causality. It comes as no surprise that a consistent orientation can easily be constructed once a causal order is given.

Theorem 33 $\boxed{\text{NDE}}\boxed{\text{COI}}\boxed{\text{LII}}\boxed{\text{NCC}}\boxed{\text{IMC}}\boxed{\text{LOR}}\boxed{\text{LCT}}$ Let $(<) \in$ Order.
(1) $im = (<) \cup (<)^{-1}$; **(2)** $(\lessdot) = im \cap (<)$;
(3) Exactly the relations $(\lessdot)$ and $(\lessdot)^{-1}$ are consistent orientations.
Proof See Theorems 3.3 and 3.9 of [2]. There P is defined as $x\ P\ y :\Leftrightarrow \underline{li}_X[x] \subseteq \underline{li}_X[y] \wedge x \neq y$. However, this definition is equivalent to ours due to Axiom LII. Furthermore, Axiom NDE can replace Axiom KDE in those proofs (see also [5]). $\square$

This means that a consistent orientation of an order-based CS exists and is unique up to reversal. But for other CSs this is less clear. In fact, a non-order-based CS that is not consistently orientable is displayed in Figs. 7 and 8. Justifying axioms (here Axiom COR) by counterexamples has turned out a very fruitful technique in CT that will exploited more often later on.

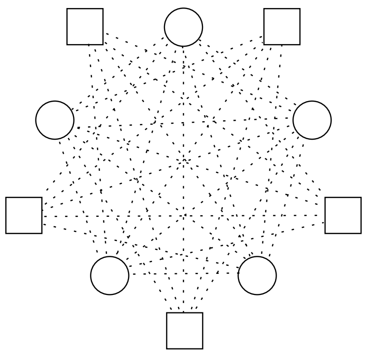

Figure 7: Not orientable, *co*

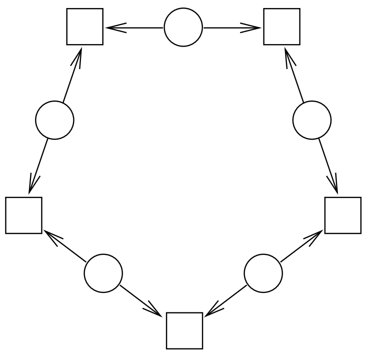

Figure 8: Not orientable, P

When we try to find a consistent orientation by applying the F-propagation rules introduced above, we may start at any *im*-arc and orient it arbitrarily. We can apply the F-propagation rules to the neighboring *im*-arc, proceeding clockwise until we return to the original arc.

But we observe that the first and the last arc we oriented violate the definition of F. The same happens when we choose the opposite orientation initially. We conclude that no consistent orientation can exist for this model. Actually, Axiom LEX is invalid here, but a larger model has been found that satisfies all basic axioms and yet cannot be consistently oriented (see [5]).

From the preceeding example we conclude that the presence of *im*-cycles of a certain type may prevent us to apply the F-propagation rules sucessfully to obtain a consistent orientation. In order to prepare a condition excluding these situations we introduce the notion of turningpoint.

Definition 34 [Turningpoints and all turningpoints]
Let $\alpha = (a_0, \ldots, a_n)$ be an *im*-chain and $\beta = (b_0, \ldots, b_n)$ an *im*-cyle.
$\mathrm{TP}(a_0, \ldots, a_n) := \{i \mid 0 < i < n \wedge a_{i-1} \ \underline{co}_X \ a_{i+1}\}$;
$\mathrm{ATP}(b_0, \ldots, b_n) := \mathrm{TP}(b_0, \ldots, b_n{=}b_0, b_1)$.

Now it is possible to exclude structures that cannot be consistently oriented by the requirement that every *im*-cycle has an even number of turningpoints. This condition can be weakened by considering only simple *im*-cycles.

ETP For every *im*-cycle α: $|\mathrm{ATP}(\alpha)|$ is even.
[Even number of turningpoints on *im*-cycles]

ETS For every simple *im*-cycle α: $|\mathrm{ATP}(\alpha)|$ is even.
[Even number of turningpoints on simple *im*-cycles]

Theorem 35 **(1)** EPT implies COR; **(2)** LCT, LOR, ETS imply ETP.

Proof The proofs are quite technical and are not given here as the main idea is very intuitive: The conditions simply ensure that applying the F-propagation rules always succeeds. That is, the rules do not lead to a contradiction as it was the case in the previous example. The complete proofs are given in [5] simplifying some proofs of [18], where the idea of turningpoints was introduced. □

For both finite and infinite lines a consistent orientation induces a very regular structure within each line.

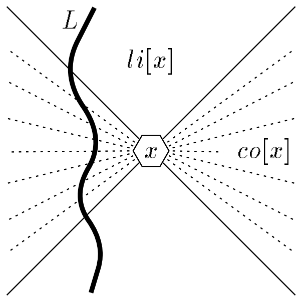

Figure 9: A coherent episode

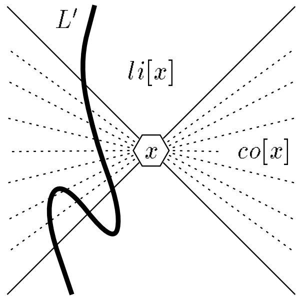

Figure 10: A scattered episode

Theorem 36 $\boxed{\text{NTR}}$ $\boxed{\text{IMC}}$ $\boxed{\text{LOR}}$ $\boxed{\text{LEX}}$ $\boxed{\text{KDE}}$ $\boxed{\text{LCO}}$
Let $L \in \text{Lines}$ and $F \in \text{Orient}$.
(1) If L is finite, there is a simple F-cycle α with $\text{Set}(\alpha) = L$.
(2) If L is infinite, there is a simple two-way infinite F-chain α with $\text{Set}(\alpha) = L$.
Proof The proof uses Thm. 26, Axiom LCO, and then Def. 10. $\square$

8 Episode Coherence

Because we defined lines as maximal cliques of li, we expect a line to represent the complete history of a signal as it propagates through time. If we take a subset of the line, we restrict ourselves to a small episode in the life of the signal.

There is an especially natural way of selecting an episode from a line: Choose $x \in X$ and consider the set $E = L \cap \underline{co}[x]$. In a sense, E contains those elements from L that are not noticed by x, because they occur concurrently to x.

In Fig. 9 an example is given where a line and an element determine an episode. But Fig. 10 looks strange. The episode is split into two parts, because the line L' enters and leaves the *co*-cone $co[x]$ twice. Hence the episode $L' \cap \underline{co}[x]$ does not constitute a continuous segment of L', thereby contradicting our intuition about lines. As we would suspect, a situation like that in Fig. 10 cannot occur if we provide a partial order by means of Axiom OBC.

Theorem 37 $\boxed{\text{COI}}$ $\boxed{\text{NCC}}$ $\boxed{\text{NDE}}$ $\boxed{\text{LOR}}$ $\boxed{\text{LCO}}$ $\boxed{\text{OBC}}$
$\forall L \in \text{Lines} : \forall x \in X : E = L \cap \underline{co}_X[x] \Rightarrow (im|_E)^*_E = E \times E.$
Proof See Satz 5.6 from [5]. Informally, if there are two unconnected segments of the episode, the elements that separate the segments must be greater or less than x in the causal order. Either way, the elements of one segment must also be greater or less than x. But they are incomparable to x by definition of E, contradiction. $\square$

But if we drop Axiom OBC, the theorem is no longer valid. In Figs. 11 and 12 we have an example of a structure where a line L and an element x form a scattered episode $L \cap \underline{co}_X[x]$. (Which two elements does it contain?) To get around this problem we formalize epsisode coherence as a possible axiom.

ECO $\forall L \in \text{Lines} : \forall x \in X : E = L \cap \underline{co}_X[x] \Rightarrow$
$(im|_E)^*_E = E \times E.$ [Episode coherence]

Theorem 38 $\boxed{\text{COC}}$ $\boxed{\text{KDE}}$ $\boxed{\text{ECO}}$ Then LCO.

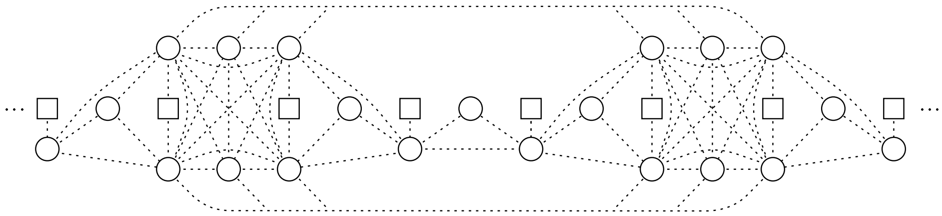

Figure 11: Scattered episodes, *co*

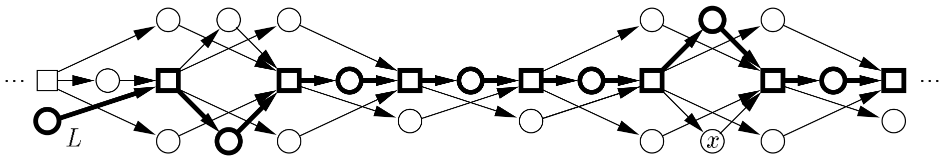

Figure 12: Scattered episodes, F

Proof See Satz 4.10 from [5]. The key idea here is to construct a set of overlapping episodes that cover the line, so that coherence can propagate through the whole line. □

Hence episode coherence is strictly stronger than line coherence. Axiom ECO is useful in the cyclic case, because scattered episodes can occur there, too, which is shown in [5], and because we cannot apply Axiom OBC. But here we will examine only its connection to Axiom LIN.

Theorem 39 $\boxed{\text{IMC}}$ $\boxed{\text{LOR}}$ $\boxed{\text{LEX}}$ $\boxed{\text{KDE}}$ $\boxed{\text{LCO}}$ $\boxed{\text{ECO}}$ $\boxed{\text{LIN}}$
Let $F \in \text{Orient}$. Then $li = F^+ \cup (F^{-1})^+$.
Proof See Satz 5.25 from [5], which requires several intricate lemmata. □

If we set $(<) = F^+$, then $(<)$ is clearly transitive. The last theorem ensures irreflexivity, hence $(<)$ is a partial order. The theorem even shows that $(<)$ is causal order. Hence Axiom ECO ensures that no information is lost when we derive a consistent orientation from an order (Thm. 33).

Corollary 40 $\boxed{\text{IMC}}$ $\boxed{\text{LOR}}$ $\boxed{\text{LEX}}$ $\boxed{\text{KDE}}$ $\boxed{\text{LCO}}$ $\boxed{\text{ECO}}$
Then LIN and COR imply OBC. □

This corollary supplements the following proposition, in the sense that Axiom ECO can be used to establish the intuitive equivalence between acyclicity (LIN) and order-based causality (OBC).

Proposition 41 $\boxed{\text{IMC}}$ $\boxed{\text{LCT}}$ $\boxed{\text{LOR}}$ $\boxed{\text{LEX}}$ $\boxed{\text{NDE}}$ $\boxed{\text{LII}}$ $\boxed{\text{COI}}$ $\boxed{\text{NCC}}$
Then OBC implies LIN and COR.
Proof Assume OBC and let $(<)$ be a causal order. Assume there is a finite line L. L is totally ordered by $(<)$. Let x be minimal in L wrt. $(<)$. Due to Axiom IMC and Axiom NTR there is a $y \in im[x]$. By Axiom LEX there is a $z \in im[x]$ with $z\ li\ y$. $F = \lessdot$ is a consistent orientation by Thm. 33. Now either $z\ F\ x \wedge x\ F\ y$ or $y\ F\ x \wedge x\ F\ z$. In any case $\exists w \in X : w \lessdot x$, contradicting minimality of x. □

This result might motivate us to restrict ourselves to order-based CT, which is fairly well understood. But by doing this we would exclude cyclic models, which are interesting, even if they pose more problems.

9 Propagating Cuts

Here we will ask whether the set of cuts can be interpreted as the set of possible global states. With the help of the propagation rules for concurrency it is possible to prove that conceiving a cut C_2 as a marking, every predecessor C_1 and successor C_3 of C_2 is also a cut. This propagation rule for cuts shows the soundness of the usual token game for the dynamics of CSs. We arrive at this rule (Prop. 43) with the help of the following lemma.

Lemma 42 $\boxed{\text{LEX}}\boxed{\text{KDE}}\boxed{\text{NCC}}\boxed{\text{IMC}}$
Let $F \in \text{Orient}$, $t \in T$, $t \in C_2$, $C_1 = C_2 - \{t\} \cup {\bullet t}$, $C_3 = C_2 - \{t\} \cup t{\bullet}$.
Then $C_1 \in \text{Cliques}(\underline{co}_X) \Leftrightarrow C_2 \in \text{Cliques}(\underline{co}_X) \Leftrightarrow C_3 \in \text{Cliques}(\underline{co}_X)$.
Proof Note that Prop. 31 and $F \in$ Orient ensure LCT and LOR. We only prove $C_1 \in \text{Cliques}(\underline{co}_X) \Leftrightarrow C_2 \in \text{Cliques}(\underline{co}_X)$.
Define $C := C_2 - \{t\}$. First we prove: $C_2 \in \text{Cliques}(\underline{co}_X) \Rightarrow C_1 \in \text{Cliques}(\underline{co}_X)$. Assume $C_2 \in \text{Cliques}(\underline{co}_X)$. Then by Prop. 16 (1) we have $\forall x \in C : \forall y \in im[t] : x\ co\ y$ such that $C_1, C_3 \in \text{Cliques}(\underline{co}_X)$. Now we show: $C_1 \in \text{Cliques}(\underline{co}_X) \Rightarrow C_2 \in \text{Cliques}(\underline{co}_X)$. Assume $C_1 \in \text{Cliques}(\underline{co}_X)$. Then for every $x \in C$ we can apply Prop. 22 (2), since the condition $\forall y \in {\bullet t} : x\ co\ y$ is satisfied (and Prop. 32 holds). So we infer $\forall x \in C : x\ co\ t$ and conclude that $C_2 \in \text{Cliques}(\underline{co}_X)$. □

Proposition 43 $\boxed{\text{LEX}}\boxed{\text{KDE}}\boxed{\text{NCC}}\boxed{\text{IMC}}$
Let $F \in \text{Orient}$, $t \in T$, $t \in C_2$, $C_1 = C_2 - \{t\} \cup {\bullet t}$, $C_3 = C_2 - \{t\} \cup t{\bullet}$.
Then $C_1 \in \text{Cuts} \Leftrightarrow C_2 \in \text{Cuts} \Leftrightarrow C_3 \in \text{Cuts}$.
Proof Note that Prop. 31 and $F \in$ Orient ensure LCT and LOR. By Lemma 42 it is already clear that $C_1 \in \text{Cuts} \Rightarrow C_2$ is a clique of $\underline{co}_X$ and $C_2 \in \text{Cuts} \Rightarrow C_1$ is a clique of $\underline{co}_X$ such that all we have to prove is the maximality of C_2 and C_1. We start with $C_1 \in \text{Cuts} \Rightarrow C_2 \in \text{Cuts}$. Assume C_2 is not maximal with respect to co. Then $\exists C_2' \in \text{Cuts} : C_2 \subsetneq C_2'$. With $C_1' := C_2' - \{t\} \cup {\bullet t}$ applying Lemma 42 yields that C_1' is a clique of $\underline{co}_X$. Obviously, we have $C_1 \subsetneq C_1'$ but this contradicts $C_1 \in \text{Cuts}$. Similarly, we show: $C_2 \in \text{Cuts} \Rightarrow C_1 \in \text{Cuts}$. Again, assume $\exists C_1' \in \text{Cuts} : C_1 \subsetneq C_1'$. Define $C_2' := C_1' - {\bullet t} \cup \{t\}$. Lemma 42 implies that C_2' is a clique of $\underline{co}_X$. Now $C_2 \subsetneq C_2'$, which is not consistent with $C_2 \in \text{Cuts}$. This completes the proof of the first equivalence $C_1 \in \text{Cuts} \Leftrightarrow C_2 \in \text{Cuts}$. Exchanging ${\bullet t}$ and $t{\bullet}$ leads to the proof for $C_2 \in \text{Cuts} \Leftrightarrow C_3 \in \text{Cuts}$. □

Now we can easily obtain the result that the case class induced by some cut (interpreted as a marking) is completely contained in the set of cuts.

Theorem 44 $\boxed{\text{LEX}}\boxed{\text{KDE}}\boxed{\text{NCC}}\boxed{\text{IMC}}$ Let $F \in \text{Orient}$, $C \in \text{Cuts}$, and $\mathcal{C}$ a case class of the net (S, T, F) containing C. Then $\mathcal{C} \subseteq \text{Cuts}$.
Proof Let $C' \in \mathcal{C}$. Then $C\ R\ C'$ and there is a $((\rightarrow) \cup (\rightarrow)^{-1})$-chain $(C, \ldots, C')$. Now we can repeatedly apply Prop. 43 to prove that $C' \in \text{Cuts}$. □

It might be a suprise that even for finite CSs it is not generally true, that the set of cuts coincides with the set of cases of some case class. There do exist CSs, where the notion of cut does not adequately capture the notion of global state, but merely the weaker notion of "conceivable" global state. In other words, although the token game in a CS is sound, it is not complete, as it does not generate the set of all cuts from a single (initial) cut. This will be demonstrated by a counterexample in the subsequent section.

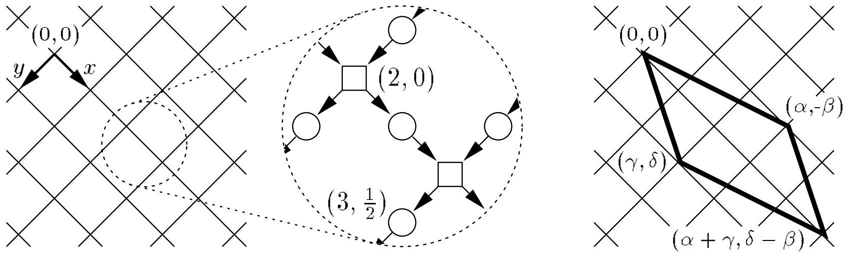

Figure 13: Standard grid and cycloid with $\alpha = 3$, $\beta = 1$, $\gamma = 2$, and $\delta = 1$

10 The Standard Grid and Cycloids

The standard grid, which is due to Petri, is the basis for the construction of a large class of CSs, namely the cycloids which were described in [14]. Its net representation is depicted in the left half of Fig. 13.

Definition 45 [Standard Grid]
The causal net (S_{SG}, T_{SG}, F_{SG}) with $S_{SG} := \{(x, y+\frac{1}{2}), (x+\frac{1}{2}, y) \mid x, y \in \mathbb{Z}\}$, $T_{SG} := \{(x, y) \mid x, y \in \mathbb{Z}\}$, $F_{SG} := \{((x, y), (x, y+\frac{1}{2})), ((x, y), (x+\frac{1}{2}, y)), ((x, y-\frac{1}{2}), (x, y)), ((x-\frac{1}{2}, y), (x, y)) \mid x, y \in \mathbb{Z}\}$ is called the *standard grid.*

As the standard grid is a causal net, we can associate a CS with it. Except for K-density, all basic concurrency axioms and also OBC and COR are satisfied. The violation of K-density is witnessed by the line $L = \{\ldots, (-1, 0), (-\frac{1}{2}, 0), (0, 0), (\frac{1}{2}, 0), (1, 0), \ldots\}$ and the cut $C = \{\ldots, (-1, \frac{1}{2}), (0, \frac{1}{2}), (1, \frac{1}{2}), \ldots\}$ which do not intersect.

As in [14] the cycloids arise from a folding of the standard grid into a cyclic net. The folding is controlled by only four natural numbers α, β, γ, and δ. By supplying an initial marking we can use Def. 5 to obtain *co* and *li*.

Definition 46 [Cycloid]
For any $\alpha, \beta, \gamma, \delta \geq 1$ the *cycloid* $(\alpha, \beta, \gamma, \delta)$ is a net (S, T, F) together with a marking M_0 defined with the help of an equivalence $(x, y) \equiv (u, v) :\Leftrightarrow \exists m \in \mathbb{Z} : \exists n \in \mathbb{Z} : x = u + m\alpha + n\gamma \wedge y = v - m\beta + n\delta$ as follows: $S := S_{SG}/(\equiv)$, $T := T_{SG}/(\equiv)$, $F := F_{SG}/(\equiv)$, and $M_0 := \{[(\langle \alpha z \rangle, \langle -\beta z \rangle)]_{(\equiv)} \mid \alpha z \in \mathbb{Z} \vee -\beta z \in \mathbb{Z}, z \in \mathbb{R}\}$, where $\langle z \rangle := z$ if $z \in \mathbb{Z}$ and $\lfloor z \rfloor + \frac{1}{2}$ otherwise.

Cycloids are regular CSs that have many interesting properties despite their apparent simplicity. With computer support we have analyzed a large class of cycloids: There are cycloids satisfying all basic concurrency axioms, but there are also cyloids, where K-density (e.g. $(2, 2, 2, 1)$) or even NCC, LCT, and LOR (e.g. $(3, 1, 2, 1)$) are violated.

Even the minimal model of CT from Fig. 1 is isomorphic to a cycloid, namely $(1, 2, 1, 2)$ as noted by Petri.

The cycloid $(4, 4, 2, 2)$ satisfies the standard axioms, but it has been verified that the case class is different from the set of cuts. There are even cuts that are dead when interpreted as a marking. The cycloid $(6, 6, 2, 2)$ has the remarkable property, that its net has three case classes inducing the same *co*-relation and therefore leading to the same CS. Hence a CS can be seen as an abstraction of the synchronization graph together with its case class.

11 Reachability of Cuts

Here some partial results concerning the identification of cuts and cases are given. The results are partial in the sense, that they are only sufficient but not necessary to gurantee that the set of cuts constitutes a case class. Note that two new axioms are assumed here.

FDE $\forall F \in \text{Orient} : \forall F\text{-cycles } \alpha : \forall C \in \text{Cuts} : \text{Set}(\alpha) \cap C \neq \varnothing$. [F-density]

CCF $\forall x \in X$: $co[x]$ is finite. [*co*-cone finiteness]

Definition 47 [Common *co*-cone]
$CO(K) := \varnothing$ if $K = \varnothing$ and $\bigcap\{co[x] \mid x \in K\}$ otherwise.

LIF $\forall F \in \text{Orient} : \forall K \in \text{Cliques}(\underline{co}_X) : \forall x, y \in CO(K) : x\ li\ y \Rightarrow$
$\exists(F|_{CO(K)})\text{-chain } (a_0, \ldots, a_n) : (x = a_0 \wedge y = a_n) \vee (y = a_0 \wedge x = a_n)$.
[li is determined by F within $CO(K)$]

COF $\forall F \in \text{Orient} : \forall K \in \text{Cliques}(\underline{co}_X) : \forall x, y \in CO(K) : x\ \underline{co}_X\ y \Rightarrow$
$\neg\exists(F|_{CO(K)})\text{-chain } (a_0, \ldots, a_n) : n \geq 1 \wedge x = a_0 \wedge y = a_n$.
[co is determined by F within $CO(K)$]

Proposition 48 $\boxed{\text{LII}}$ $\boxed{\text{COI}}$ $\boxed{\text{COC}}$ $\boxed{\text{LIC}}$ $\boxed{\text{KDE}}$ $\boxed{\text{NCC}}$ $\boxed{\text{LCO}}$ $\boxed{\text{LEX}}$ $\boxed{\text{CCF}}$ $\boxed{\text{COR}}$
Then FDE, COF, and LIF imply $\forall C, C' \in \text{Cuts} : C\ R\ C'$.

Proof The proof carried out in [18] for S-cuts, i.e. cuts containing states, can be easily adapted. □

Theorem 49 $\boxed{\text{LII}}$ $\boxed{\text{COI}}$ $\boxed{\text{COC}}$ $\boxed{\text{LIC}}$ $\boxed{\text{KDE}}$ $\boxed{\text{NCC}}$ $\boxed{\text{LCO}}$ $\boxed{\text{LEX}}$ $\boxed{\text{CCF}}$ $\boxed{\text{COR}}$
Let $F \in \text{Orient}$, $C \in \text{Cuts}$, and $\mathcal{C}$ a case class of the net (S, T, F) containing C. Then $\mathcal{C} = \text{Cuts}$.

Proof Combine Thm. 44 and Prop. 48. □

Obviously, a case class that is also a subset of cuts is necessarily safe and even secure. It is additionally proper, if it encompasses all cuts. A case class that is not identical to the set of cuts may nevertheless induce a *co*-relation generating this set of cuts (as it is true for the cycloid $(6, 6, 2, 2)$). But there may be models of CT where no single case class can completely generate *co*, as some results in the context of cyclic orders (see below) suggest, although Axiom COR and the basic concurrency axioms are satisfied.

12 Conclusion

The present work can be seen as an attempt to explore CT without assuming any underlying partial order. Most of the difficulties encountered are due to this general approach. Nevertheless many interesting properties can be derived from the basic axioms alone. Furthermore, we worked out some of the problems and offered solutions in terms of additional axioms. In particular we addressed the possibility of broken lines and scattered episodes and the problem of ensuring consistent orientability by local assumptions. Furthermore, we found that the desirable identification of cuts and cases is not valid in general, not even for finite structures. Some sufficient conditions have been proposed to ensure this property.

We know that Petri is aware of the difficulties arising with cyclic CSs ([13]). Already in [8] the necessity of some kind of cyclic orders for the axiomization of cyclic CSs was emphasized. There Petri suggested the concept of separation quadruples as a formalization of cyclic orders. A separation quadruple $\{\{a,c\},\{b,d\}\}$ represents the intuitive fact that elements a and c separate b and d on some line. However, counterexamples show that the representation of concurrency structures as sets of separation quadruples is not really helpful to ensure consistent orientability, because separation quadruples are conceptually unoriented, similar to the li-relation of CSs. (Directed) cyclic orders (as formalized in [19]) might provide a local solution to consistent orientability. They even allow us to speak about the stronger property of global orientability, the possibility to extend a (partial) cyclic order to a total one, a property, which is difficult to formulate in terms of li and co alone, but seems to be necessary for an identification of cases and cuts.

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The works [5], [18], and [19] by the authors are accessible via WWW at the URL `http://www.informatik.uni-hamburg.de/TGI`.

A Notation and Basic Definitions

Definition 50 [Binary relations]
Let R, S be binary relations. The *relational composition* of R and S is $R \odot S := \{(x,z) \mid \exists y : (x,y) \in R \wedge (y,z) \in S\}$. The *image* of a set Y under R is $R[Y] := \{z \mid \exists y \in Y : y \; R \; z\}$. The *reflexive closure* is $\underline{R}_Y := R \cup \mathrm{id}_Y$. The *transitive closure* is denoted by R^+. R^*_Y is the *reflexive, transitive closure* $R^+ \cup \mathrm{id}_Y$. R is *acyclic*, iff R^+ is irreflexive. Let R be a *similarity*, i.e. reflexive and symmetric. A set Y is a *clique* of R, iff $Y \times Y \subseteq R$. A *ken* of R is a maximal clique of R wrt. set inclusion. Let R be an *equivalence*, i.e. reflexive, symmetric, and transitive. The *quotient of a set* X under R is $X/R := \{R[x] \mid x \in X\}$. The *quotient of a relation* R' under R is $R'/R := \{(R[x], R[y]) \mid (x,y) \in R'\}$. Let R be any relation. A sequence α is an *R-chain*, iff $\alpha_i \; R \; \alpha_{i+1}$ for all indices $i, i+1$ of α. A *simple R-chain* is an R-chain with $a_i \neq a_j$ for all indices $i \neq j$. For an R-chain $\alpha = (a_0, \ldots, a_n)$ let $\mathrm{Set}(\alpha)$ denote the set $\{a_0, \ldots, a_n\}$. An R-chain $(a_0, \ldots, a_n)$ is an *R-cycle*, iff $n \geq 1$ and $a_0 = a_n$. A *simple R-cycle* is an R-cycle $(a_0, \ldots, a_n)$ with $\forall i, j \in \{0, \ldots, n-1\} : i \neq j \Rightarrow a_i \neq a_j$. A binary relation $(<)$ is a *(strict) partial order*, iff $(<)$ is irreflexive and transitive. In this case $(\lessdot) = (<) - (<)^2$ is the *immediate predecessor relation.*

Definition 51 [Nets]
A *net* N is a triple (S, T, F) with $S \cap T = \varnothing$, $S \cup T \neq \varnothing$, $F \subseteq (S \times T) \cup (T \times S)$ and $\mathrm{dom}\,(F) \cup \mathrm{ran}\,(F) = S \cup T$. S, T, and F are the set of *states*, the set of *transitions*, and the *flow relation*, respectively. $X := S \cup T$ is the set of *net elements*. Let N be a net and x a net element. The *preset* and *postset* of x are ${\bullet}x := F^{-1}[x]$ and $x{\bullet} := F[x]$, respectively.