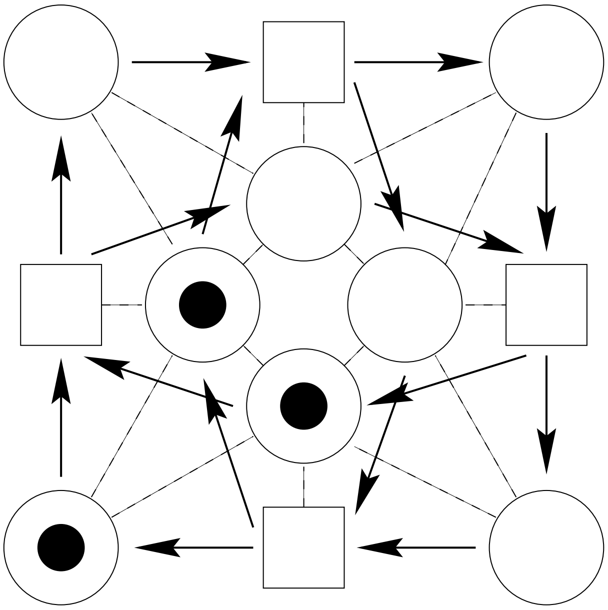
# Concurrency Theory

## of Cyclic and Acyclic Processes



Mark-Oliver Stehr
Universität Hamburg
Fachbereich Informatik
Arbeitsbereich Theoretische Grundlagen der Informatik

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Mark-Oliver Stehr
Universität Hamburg
Fachbereich Informatik - TGI
Vogt-Kölln-Str. 30
22527 Hamburg
e-mail: stehr@informatik.uni-hamburg.de

## Abstract

This work investigates the axiomatic concurrency theory proposed by Carl Adam Petri as a basis of general net theory starting with physically motivated axioms. A formulation in terms of partially ordered sets is intensionally not adopted here, in order to deal with this theory in a more general setting, viewing causality and concurrency as pure similarity relations. Concurrency structures, which are the models of this theory, are intended to describe the synchronisation structure of possibly cyclic processes at an arbitrary level of abstraction.

The major result of this work is that under certain conditions we can associate exactly two nets (of which one is the inverse of the other) with every concurrency structure. An appropriate elementary-net-specification based upon one of these nets has a case class that coincides with the class of statelike cuts. In other words, under appropriate assumptions supplementing Petri's axioms the token game is sound and complete to evolve the dynamics of concurrency structures.



## Kurzfassung

Diese Arbeit untersucht, die von Carl Adam Petri vorgeschlagene, axiomatische Concurrency-Theorie als Basis der allgemeinen Netztheorie, ausgehend von physikalisch motivierten Axiomen. Eine Formulierung mit Hilfe von partiellen Ordnungen wird absichtlich vermieden, um die Theorie auf einer allgemeineren Grundlage zu studieren, die Nebenläufigkeit und Kausalität als reine Ähnlichkeitsrelationen auffaßt. Die Concurrency-Strukturen, die sich als Modelle dieser Theorie ergeben, sollen die Synchronisationsstruktur von möglicherweise zyklischen Prozessen auf einer beliebigen Abstraktionsebene beschreiben.

Das Hauptresultat dieser Arbeit ist, daß wir unter bestimmten Bedingungen genau zwei Netze (ein Netz und sein Inverses) mit jeder Concurrency-Struktur assoziieren können. Ein geeignetes elementares Netzsystem, das auf einem dieser Netze basiert, hat ferner eine Fallklasse, die mit der Klasse der zustandsartigen Schnitte identisch ist. Mit anderen Worten: Das übliche Markenspiel ist unter geeigneten, Petris Axiome ergänzenden Annahmen, korrekt und vollständig, um die Dynamik von Concurrency-Strukturen zu entwickeln.

# Contents

# 1 Motivation

Is it true that between two points of time (space) there is always another one? Does the point of time (space) $\sqrt{2}$ really exist? The first question addresses the adequacy of modelling an analogous quantity by the set of rational numbers (equipped with the conventional total order) instead of using a scale of integers. The fundamental difference between integers and rationals is that rationals do not contain any jumps as they are dense. The second question goes even a step further appealing to the property of the totally ordered set of rational numbers to contain gaps. The usual way to fill these gaps is to choose the smallest completion which are the real numbers to describe analogous quantities.

From the viewpoint of measurements, which can only supply a finite amount of information in a finite interval of time, it will certainly never be possible to find any evidence, which could solve our problem in favour or against the assumption that the intrinsic character of some physical property is actually a real number that is not rational. So for our measurement scale it is sufficient to employ the rationals. Real numbers, which are conventionally used to represent physical quantities, are insofar convenient as they allow the use of a rich and mathematically well-elaborated library of analytic methods (namely those for partial differential equations) to predict the time evolution of dynamical systems. On the other hand only few complex problems can be solved analytically, that is, in such a way that one can derive the exact solution in a finite number of steps. So it is common practice to solve computational problems by numerical approximations. Again a calculation yields always a finite amount of information. So rationals are certainly sufficient to represent the result with arbitrary accuracy.

But even with the rational model we have a paradoxical situation when we try to describe smooth movements. Certainly the physical motion of an object on some scale modelled by rationals should skip no element on its path. Intuitively it should visit the elements one after the other. But, as every rational interval consists of an infinite number of elements, the moving object will never leave this interval, which is certainly a contradiction with our experience. The conventional solution is that the position of our object must be a function of a rational time scale. But this only postpones the actual problem to another quantity, which is the time in this case: Is it not true that starting at a finite interval of time every point is visited at some instant? Again this implies that an infinite number points have to be visited one after the other, which is intuitively not reconcilable with the boundedness of the interval. From a different point of view we can reformulate the problem as follows: Given a rational it is impossible to find an immediate successor, such that the temporal evolution of a physical system necessarily stops (as it does not know where to go). But according to our experience this does not happen.

A radical answer to this paradox is to reject even the rationals as valid representatives of analogous quantities and to go back to the integers where we know that every interval is actually finite. This leads immediately to the mathematical theory of discrete dynamical systems which has a counterpart in computer science, namely, in the theory of cellular automata. In fact, recently there have been many (although non-concrete) proposals to take information (in form of binary decisions) as the most fundamental concept from which every other aspect of physical existence has to be derived (see *Wheeler 1990*). Following this line physical systems could be simply viewed as information processing systems.

This preceding solution to the dilemma of analogous quantities is simply to deny their existence with the assumption that every physical property can be described by digital quantities. But now we are faced with the problem that the digital representation must be exact to derive sound results, that is, we have to consider every detail of the problem domain, what is impossible as the discrete nature (if it really exists) of physical quantities is a very fine one and exact measurements are impossible due to practical and theoretical restrictions. Moreover the solution to represent every detail exactly by digital quantities and to reformulate microscopic physical laws on that level that is believed to be the atomic one leads to a mismatch between theory and application as most of the real-world problems have to be solved on a macrosopic level.

An alternative approach towards a solution of these problems might be given by concurrency theory, which was proposed by C.A.Petri (e.g. in *Petri 1987*) as a general theory to deal with uncertainty of analogous quantities, although this work concretely refers to space and time. With the more general interpretation of concurrency theory as a theory of measurement (*Smith 1989*) the two constituting relations of causality and concurrency can be viewed as the relations of comparability and indifference, respectively. For total orders (e.g. the standard models for measurement scales: integers, rationals and reals) every two elements are comparable. The explicit articulation of indifference (which arises from the impossibility to compare two analogous quantities due to theoretical or technical restrictions) is a major aspect of concurrency theory. The essential idea of the time-space interpretation is to identify physical laws that are valid on every conceivable level of abstraction such that the incompleteness of the current view of a system is accepted as a natural restriction. The question if the actual nature of analogous quantities must be described by integers, rationals or reals is meaningless in this theory, as it allows us to adopt that view that is adequate in the context of a particular problem. The general requirement to express uncertainty introduces a further dimension into the theory, that is orthogonal to the measured quantity. Applying this idea to the time-space interpretation the existence of space (or concurrency) is a necessary consequence of the temporal dimension (or causality). On the other hand time is a necessary resource to explore the spatial dimension.

Before we step deeper into the matter an overview will be given of cellular automata to determine differences and similarities between these two approaches.

## 2 Cellular Automata and Discrete Physics

A computational approach to discrete physics is based on Cellular Automata (*Feynman 1982; Toffoli 1988*) proposed by von Neumann originally to analyze the capability of self-reproduction within cellular space. Interestingly they also provide a very natural and uniform model of parallel computing. A cellular automata can bee seen as an infinite, regular network (typically an n-dimensional grid) of cells each of them being an identical finite automata with the capability to communicate with a finite neighborhood. A local rule determines the successor state of an automaton given the states of all cells in its neighborhood. A configuration of a cellular automaton ( the global state) describes the individual local state of each cell. Cellular automata evolve in discrete time steps where a new configuration is always obtained by applying the local rule to all cells synchronously.

Cellular automata seem to be appropriate for the description of temporally and spatially discrete dynamical systems respecting the locality principle, as the speed of information flow within the Cellular Automata is limited. The physical state has to be coded into the local state of each cell and global conservation of certain quantities can be realized by the appropriate choice of the local rule.

Cellular automata are deterministic and may be irreversible since information about the past may get lost if two configurations are mapped to the same successor. Microscopic reversibility of physics (which is also necessary for quantum mechanics of closed systems) can be modeled with reversible cellular automata which are forward- and backward-deterministic by definition (*Toffoli und Margolus 1990*). An important result in the field of reversible computation is the existence of universal, reversible machines.

Identifying cellular space-time with physical space-time has led to interesting applications in the field of gas and fluid dynamics. Nice theoretical results presented in *Toffoli 1990* (although in a more general setting) are that (statistical) variational principles known from analytical mechanics can be derived from very weak assumptions (e.g reversibility) and that statistically a microscopic grid is capable to show full rotational symmetry on the macroscopic level. Furthermore there are some basic ideas how Lorentz-invariance could emerge as a statistical feature.

Associating the global simulation time with the local time for actual events within the cellular automata may be adequate at low velocities, but Relativity Theory requires a different time-space-metric for each inertial system moving in the cellular automata, which is determined by the Lorentz-transformation.

New approaches, in particular Fredkin's program of Digital Mechanics (see *Fredkin 1990*), try to go a step further. The naive correspondence between physical space-time and cellular space-time is given up, and time is made explicit by assigning it to an additional dimension. This leads to a notion of information-cones within cellular space. These are exactly those regions relative to a particular cell which may have an influence on its successor state. The explicit representation of time within the formalism is also an essential feature of concurrency theory.

Unfortunately, so far there is no competitive approach of discrete physics, which is indeed an alternative to conventional theories. Actually, discrete physics suffers from several inconveniences arising from lacking mathematical methods for the treatment of dynamical systems, as the dynamics is not continuous in the sense of analysis. In contrast to the cellular automata approach concurrency theory leads to a generalized concept of continuity that allows countable and even finite sets to be continuous. Of course a closed and applicable mathematical theory (e.g. a counterpart to analysis) does not yet exist, although there are promising connections to the field of topology, which is also the modern basis of analysis.

As a motivation for the need to investigate a different theory, some hints are presented justifying the view that (conventional) cellular automata are incapable of the exact simulation of physical systems and are furthermore not adequate, also from an application-oriented point of view.

- A first point is the global synchronization of all cells. There is no evidence that this synchronization really exists in nature and furthermore it is not even necessary if the local time differs from simulation time by explicit simulation of local clocks. Asynchronous cellular automata may provide a solution to this problem.

- Moreover, cellular automata are deterministic. Even if we assume physical determinism, this excludes the separate analysis of subsystems or macroscopic views which are not necessarily deterministic due to environmental influence. So either we can only describe closed systems without environment, or we have to choose an approach using nondeterministic cellular automata.

- Nondeterminism emerges not only from unsolved alternatives but may also occur due to concurrency. Concurrent nondeterminism arises from the fact that it is impossible to define an objective order of events, which are spatially separated from each other, such that the exchange of signals between them is impossible. How is it possible to realize this kind of nondeterminism with cellular automata? And how is it possible to separate these two aspects of nondeterminism?

- The vision of an exact simulation of all physical phenomena with cellular automata depends on the general assumption that a discrete and exact representation of physics is possible and that the actual atomic units of physical quantities (e.g. the smallest units of time and space) and corresponding local rules can be found.

- Even if an appropriate description of microscopic physical laws could be found in terms of a local rule for cellular automata, the applicability is not guaranteed, as the exact simulation, which has to be carried out on the atomic level will be practically infeasible from the computational point of view. Although a straightforward realization of parallel computers on the basis of cellular automata is conceivable, the enormous number of elements, which are necessary to represent the states of a small volume element of interest, can certainly not be realized with current technology.

- If an appropriate cellular automata can be found, this is certainly of interest for the theoretical foundation of physics. On the other hand, if this cellular automata can be actually realized, the practical value of an exact simulation is not obvious. Firstly the initial conditions are mostly not known exactly, and secondly exact results covering all microscopic details are often not necessary for practical applications. Unfortunately the local rules are only valid on the atomic level and a naive reduction of the resolution of the cellular automata (by reducing the number of cells and states per cell) leads to incorrect results. Altogether we recognize the general necessity for a means bridging the gap between different levels of abstraction which has not yet been developed for cellular automata.

- A major problem of cellular automata, which is deeply connected to the previous argument, is the inherent discontinuity in the theory. There was no attempt to find a solution to the apparent contradiction between discreteness and continuity. Concretely, the configuration between two time-steps is not defined, there are jumps between configurations due to jumps on the total order of integers, which has been chosen as a model of time. Of course this is also true for all other quantities represented within the cellular-space. At least on higher levels of abstraction this way of dealing with analogous quantities is inadequate to describe smooth changes that are part of our everyday experience.

- Again seen from the application point of view the regular grid-structure of cellular automata is often inappropriate for the problem domain. The structure of the problem has to be coded artificially into the states and the local rule. In other words, to apply cellular automata the non-uniform structure of a practical problem has to be mapped to the uniform formalism of cellular automata. As a consequence, the lacking flexibility of cellular automata might lead to unnecessarily complex representations of an originally simple problem. Certainly this point is linked with the previous ones, as it is the right abstraction which is a primary ingredient to solve practical problems.

Comparing cellular automata and Net Theory these two approaches are in a certain sense articulations of two different extremes: The cellular automata formalism is based on structural uniformity and puts the whole complexity into its local rule. In the formalism of nets the local rule (which is the token game) is quite simple and the structure of a net is used to represent the complexity of the problem. As this work only deals with concurrency theory, it will not be possible to provide a solution to all of the problems mentioned above. The preceding list should be taken as a motivation for the alternative approach of Net Theory which includes concurrency theory as an essential component. We will mainly address the points concerned with causality and concurrency. In particular, concurrency theory does not assume an apriori global synchronization. We will not deal with those problems connected with the notion of state space, as they cannot be captured by the notion of concurrency and have to be examined in a more general theory. Concerning continuity it is believed that a solution of this issue can be provided by concurrency theory (as it was indicated in *Petri und Smith 1987*), although a detailed analysis of this topic is beyond the scope of this work. Finally we come to the question how to change between different levels of abstraction of the same system. Once we have established a connection between concurrency theory and the formalism of nets (and this will be done in this work), topological methods (that can be found in *Fernández 1975*) can be applied to describe continuous mappings between different views.

# 3 Basic Concepts

General Net Theory is an attempt to combine the different special applications and formalisms concerning nets into a uniform framework (*Petri 1980b*). It can be imagined as a hierarchy containing a theory of concurrency and possibility on the lowest level followed by a theory of nets, elementary net systems and information flow. Higher level net formalisms and specializations constitute the upper levels of General Net Theory.

On a tutorial at Milano in April 1989 C.A.Petri presented a “Combined Axiomatics for Concurrency, Causality and Possibility” as a basis for General Net Theory. He also presented these ideas in lectures at the University of Hamburg (*Petri 1988a*, *Petri 1988c* and *Petri 1989*). The major aim was to give a justification for Net Theory based on a combinatorial formulation of physical laws. Unfortunately work on this axiomatic system has not been finished yet, but nevertheless there are some interesting parts which have already been published (*Petri 1980a; Petri 1982; Petri 1987*). Concurrency theory can be conceived as a projection (or specialization) of the combined axiomatics including causality and concurrency but excluding possibility. To exclude possibility means that alternatives do not occur. So we might imagine the structures of concurrency theory as describing exactly one possibility of the system's evolution. In addition to approaches using partial orders to describe the causality structure of processes (thoroughly developed in *Best und Fernández 1988*), concurrency theory (as presented in *Petri 1980a* and *Petri 1987*) allows spatially and temporally cyclic structures such that infinite, repetitive processes can be described by finite means. In the following a survey will be given of the pragmatic ideas and fundamental physical concepts relevant in General Net Theory.

## 3.1 Pragmatic Ideas

In order to get a rough impression how General Net Theory tries to capture apparently incompatible phenomena of very different disciplines some general ideas will be mentioned that might play a role in the development of a complete theory of systems. According to *Petri 1988b* the carrier of the theory should be a (possibly infinite) set $X$ of pragmatic units equipped with four symmetric and irreflexive binary relations: Causality ($li$), Concurrency ($co$) and Alternative ($al$), Togetherness ($to$).

For a given system the elements of $X$ are articulations of desired or observed behavior. They are called pragmatic units, as they are those elements, which have to be considered in a certain pragmatic context. This means that it depends on the problem domain and the desired results how detailed the concrete problem is represented. In other words, the degree of accuracy is determined by the application and not by the theory.

The causality relation ($li$) holds between those elements that are causally dependent of each other. This means one element has a definite effect upon the other one. The direction of the effect is not essential here. Two elements are concurrent ($co$), if both of them occur but without influencing each other. Notice that concurrency does not necessarily imply simultaneity. Two elements are alternative ($al$), if they are mutually exclusive, that is the occurrence of one of them excludes the occurrence of the other one. The relation of togetherness ($to$) introduces subjective (that is observer-dependent) aspects into the theory. Different observers might use a different words to denote elements of $X$, which are objectively indistinguishable. Elements related by $to$ are coincident, that is, they occur at the same place and time and in the same state. As $to$ is an equivalence relation, it is possible to consider the objective quotient structure $(X/to, li/to, co/to, al/to)$ proceeding without $to$. This is the reason why we are not concerned with $to$ in this work.

Apart from the requirement that $li$, $co$ and $al$ are symmetric and irreflexive, a major assumption (the completeness axiom) is that between every pair of different elements we can establish $li$, $co$ or $al$. This means we can express the relation between every two elements within the theory. A further elementary requirement is that $li$ and $co$ as well as $li$ and $al$ are mutually exclusive. Intuitively, it is clear that causality and concurrency are not reconcilable and, of course, two elements which are mutually exclusive ($al$) cannot be causally dependent of each other. Further axioms concerning $al$ are necessary, but as it was mentioned above only that part of the theory dealing with $li$ and $co$ is worked out yet. Nevertheless the relation $al$ will not be ignored in the following sections in order to give an idea into which direction concurrency theory might have to develop.

Certainly we cannot deny the relevance of these relations on the low level of microscopic physics (e.g between particles) as well as on the high level of planning and organization (e.g. between persons, groups). An essential point concerning the axioms of the desired theory is their validity on every conceivable level of abstraction. This is a strong requirement, as not every physical law can be expressed in a form, which remains invariant, if the current view of the system is changed.

Once a theory is formulated in terms of causality, concurrency and alternative, an adequate formalism is required to deal with practical problems. It might turn out that the formalism of nets and elementary net systems is appropriate, as it allows us to describe causality, concurrency and alternatives in a natural manner. The major aim of this work is to show the adequacy of elementary net systems for a theory restricted to causality and concurrency.

## 3.2 Causality and Time

From Relativity Theory we know that time is not absolute between moving inertial systems. Observers in different inertial systems record events with respect to their own subjective time-space-metrics. But if we apply the fact of limited signal propagation velocity consequently, we find that we have even less knowledge about the time at different places within one inertial system: Consider one inertial system with two clocks at different places A and B. Both clocks are at rest. Now we would like to know if time within the inertial system is the same at A and B. Clearly we cannot decide this with arbitrary accuracy, since all signals which may be exchanged between A and B are limited by the speed of light. What is typically done is to postulate that time is the same at A and B (this can never be refuted). That time is absolute within one inertial system leads to the convenient Lorentz-transformation, which is applied between inertial systems. In Net Theory it is not even postulated that time is objective within one inertial system. As a theory based on objective concepts is prefered, one can argue that time is inadequate as a basic notion and a theory formulated in terms of pure causality is justified.

So the notion of time in General Net Theory is not an elementary one. According to Petri, time is nothing more than the state of clocks. So time has to be modeled explicitly by clocks within the system, which are synchronized with each other. In this sense time is treated as every other analogous quantity that could be measured. As we have seen, clocks which are spatially separated may show different times, even if they are synchronized. The formal method to deal with this in Net Theory is based on the concept of synchronic distance determining the degree of synchronization between these clocks (which might depend on the spatial distance between them). For observers within the system it is impossible to find an experiment that could be carried out to detect the difference in time at different locations, as every exchange of signals forces the two clocks to resynchronize.

If time or temporal properties are mentioned in the subsequent sections, this only refers to the objective aspect of causality but not to subjective delays between events.

A major result of this work is that under certain conditions exactly two choices for the arrow of time are possible of which one is the inverse of the other. Intuitively these two directions correspond to the temporal evolution of a physical system in opposite directions of time.

### 3.3 States and Locality

The global state of a system is a maximal set of elements that might coexist concurrently. Physically, the global state consists of all local states on a spacelike surface in time-space. The fact that the global state is distributed in space makes it impossible for observers within the system, which are themselves restricted by physical laws (in particular they cannot exchange signals faster than light), to exploit the global state completely on one space-like surface.

According to the principle of locality, the temporal evolution of the global state is governed by a local rule. As there is no apriori metric defined on the structure of pragmatic units, it is a major difficulty to define an appropriate notion of locality. Of course, this is one of the points, which will be addressed in this work. As the rule is a local one, a global state, that is extended in space, is developed independently at different locations such that not all events (the application of the local rule might be seen as an event) can be totally ordered (these events are also concurrent pragmatic units). So with the presence of concurrency a total order of events is the subjective impression of a particular observer.

With the presence of alternatives ($al$), we would have to deal with an additional problem, namely, the structure of a branched time scale. The temporal evolution might at a certain instant of time decide to drive the system into one of several possible directions which are mutually exclusive.

### 3.4 Determinism and Reversibility

The usual notion of nondeterminism has to be separated into two different aspects: Nondeterminism of concurrency and nondeterminism of alternatives.

Nondeterminism of concurrency occurs, if two spatially separated events which are causally independent of each other may occur in undetermined order, when they are mapped on a total time scale of some observer. In particular different observers may find different occurrence sequences of events, although the causality structure is always the same. Different observers will agree on the fact that these events occur, but they may disagree about the order of perception. The time-reversal-counterpart of concurrency is synchronization.

Nondeterminism of alternatives corresponds to the indeterminate choice between possibilities for future dynamical evolution. In contrast to concurrent nondeterminism the alternatives are mutually exclusive. Alternatives can be interpreted as sources of information. Immediately after solving the alternative the system contains the information which alternative has been chosen. Note that an alternative does not necessarily lead to nondeterminism, since the alternative may be solved within the system. If all forward-alternatives are solved within the system the system is deterministic (with respect to alternatives). The time-reversal-counterpart of an alternative is the backward-alternative, where two or more possible branches of time evolution are combined. Information contained in the system may be erased in this case. To avoid the production and erasure of information it is possible to conceive a completion of our system by some environment which solves all (forward- and backward-) alternatives such that the total amount of information is conserved. Such a system which is forward- and backward-deterministic is also called reversible, and it is exactly this reversibility which is crucial in physical systems on the microscopic level.

Notice that in concurrency theory we concentrate on nondeterminism of concurrency, but nondeterminism due to alternatives (which are essential to construct information processing systems) does not occur, as we do not have any alternatives at all.

### 3.5 Continuity

General Net Theory can be conceived as a method to deal with dynamical systems in a way involving only finite or countably infinite concepts. The major criticism of General Net Theory concerning todays physical methods (which are certainly successful, we cannot deny that) refers to the assumption that continuous change can only be achieved by means of uncountability. Here continuity means continuity of orders (also refered to by order-completeness). Continuity of total orders was axiomatized by Dedekind (a continuous order does contain neither gaps nor jumps) who applied this notion to construct the continuum of real numbers. Petri recognized that a generalization of Dedekind-continuity from total to partial orders yields combinatorial partial orders which are continuous but nowhere dense. The precise definition of generalized Dedekind-continuity (D-continuity) of partial orders can be found in *Petri und Smith 1987; C.Fernández und A.Merceron 1987; Best und Fernández 1988.*

It is not intended to deal with continuity in this work, but our choice of axioms which is not exactly the set of axioms proposed by Petri should be justified. Concurrency structures, as they will be introduced below, should describe continuous changes (movements) in time-space. D-continuity with respect to certain sets of concurrency axioms was analyzed in *Fernández und Thiagarajan 1983; Best und Merceron 1985.* The result was that the original set of axioms proposed by Petri in *Petri 1980a* does not imply D-continuity. It is the cone-intersection-property (see *Petri 1987*) and combinatorialness (of the temporal partial order) which are additionally necessary to derive D-continuity. We will see that for acyclic structures (which have to be defined) our choice of axioms implies the cone-intersection-property and ensures that lines are combinatorial.

Strictly D-continuity is only defined for partial orders. We will see that not every concurrency structure can be represented by a partial order. It is lacking some more general concept to deal with cyclic structures. With this concept it should be possible to extend the definition of D-continuity to cyclic orders (whatever this may be).

For the sake of completeness a further form of continuity is mentioned, which is crucial when different (more or less detailed) views of a system have to be combined via morphisms. If we define an appropriate topology on our system (for nets this can be done in a natural manner), we can allow exactly those functions as valid refinements which are continuous mappings with respect to this topology. Proceeding in this way we are always sure that refinements do not destroy the coherence of our structure. It is obvious that refinements are essential for modelling physical systems (e.g. distinction between microstates and macrostates).

## 4 Axioms of Concurrency Theory

Concurrency theory, as it was proposed by Petri, is the theory of concurrency structures, which can be applied to describe physically realizable, possibly cyclic processes in time-space. As it was already mentioned, a slightly modified version of concurrency theory is presented here. For a historical survey of concurrency theory comparing the different axiomatic approaches *Müller 1993* is worth to read. An overview of concurrency theory is also given in *Fenske 1992*.

A concurrency structure $CS$ is a triple consisting of a set $X$ (which may be finite or infinite) and two binary relations $li$ and $co$ defined on $X$ satisfying a variety of axioms which are given below.

**Scope S1** Let $CS = (X, li, co) \wedge li, co \subseteq X \times X$.

Although there are further interpretations of concurrency structures (see *Smith 1989*) concerning the representation and measurement of analogous quantities only the standard interpretation will be considered which is the following:

- $X$ is a set of elements in time-space. There are no a priori requirements on the nature of $X$ concerning dimensionality, cardinality or density. Furthermore no metric is assumed on $X$.
- $li$ is the causality relation. It covers our intuitive notion of the causal dependence between two elements. The direction of causal influence is ignored here, avoiding to break the temporal symmetry by introducing causes and effects on this level. Altogether we can say that $li$ establishes the temporal dimension in $X$.
- $co$ is the relation of concurrency. Two elements are concurrent, if they are spatially separated enough to exclude any interaction between them. This is a parallel to the notion of space-like distance between two points in Minkowski-Space. On higher levels of abstraction it is not necessarily only the limited speed of light, which is the source of concurrency. Concurrency may also be realized by excluding causal dependencies with the help of certain technical boundary conditions.

We will use this interpretation to give a short motivation for each of the subsequent axioms. Sometimes it will be necessary to anticipate some results that will proven in the subsequent sections.

**Axiom A1** $\mid X \mid > 1$. □

This axiom excludes trivial structures which are empty or contain only one element. In these structures there is neither causality nor concurrency. Hence they are not of interest for us.

**Axiom A2** [Completeness] $co \cup li \cup id_X = X \times X$. □

Every two distinct elements of time-space must be concurrent or causally dependent of each other. There is no further possibility for the relation between them.

**Axiom A3** [Disjointness] $co \cap li = li \cap id_X = co \cap id_X = \emptyset$. □

Causality excludes concurrency. Together with the previous axioms this establishes concurrency and causality as complementary relations. That causality and concurrency are irreflexive is assumed for mathematical convenience.

**Axiom A4** [Symmetry] $co^{-1} = co$. $\square$

Concurrency is symmetric. This corresponds to the assumption that there is no preferred direction in space which could cause an asymmetry (isotropy of space). Obviously we could equivalently require $li^{-1} = li$ which ensures the symmetry of causality. Hence there is no privileged direction of time either. So if $x\ li\ y$ holds we can say: $x$ and $y$ are causally dependent, $x$ affects $y$, $y$ affects $x$ or $x$ has an influence on $y$ and $y$ has an influence on $x$. In case of $x\ co\ y$ we simply say $x$ is concurrent to $y$ which already expresses a certain symmetry.

**Axiom A5** [Irreducibility] $\tilde{co}_X = \tilde{li}_X$. $\square$

This is the axiom of irreducibility. It is similar to the axiom of extensionality of elementary set theory. Every two distinct elements of time-space should be distinguishable in terms of their relation of concurrency as well as causality with respect to other elements: $\tilde{co}_X = id_X = \tilde{li}_X$. We will see that the apparently weaker axiom is sufficient to guarantee this property in combination with the previous axioms. Physically, it will be postulated that there is no interior structure which could lead to a further identity of elements in time-space. A structure which does not satisfy this axiom can be successively reduced by identifying those elements which cannot be distinguished in terms of concurrency or causality. But this has to be taken with care, as it might happen that other axioms get violated in this reduction process.

**Axiom A6** [Coherence] $co_X^* = li_X^*$. $\square$

The axiom of coherence requires that two elements which are connected by a finite chain of causality-steps should also be connected by a finite chain of concurrency-steps and vice versa. We will see that with this axiom it follows that for an arbitrary pair of elements there are finite causality- and concurrency-chains connecting them. It is exactly this axiom which establishes the finite character of the theory. Note that this intentionally does not exclude infinite concurrency structures.

**Axiom A7** [Finiteness of concurrent neighborhood] $\forall x \in X : co[x]$ is finite. $\square$

For every element $x$ it is postulated that there are only a finite number of elements that can coexist concurrently to $x$. This requirement corresponds to the very strong assumption that space is finite. We could argue that space may extend within the temporal evolution, but there was one instant of time where space was finite, such that space remains finite but may expand arbitrarily. We will see that many important properties of concurrency structures are connected with this axiom. A further reason to introduce this axiom is to avoid difficulties arising from infinite cuts on the level of elementary net systems.

**Definition D1** [Lines and Cuts]

a) $Lines := Kens(\underline{li}_X)$;
b) $Cuts := Kens(\underline{co}_X)$.

$\square$

Lines and cuts correspond to time-like and space-like surfaces in Minkowski-Space. A line is a maximal clique of causality also known as world-line. Each particle or signal propagates along some particular world-line. A cut is a concurrency-clique of maximal extension. It can be conceived as a spatial snapshot of our physical system and represents the global state at that instant of time relative to some observer which is not necessarily at rest.

**Axiom A8** [K-density] $\forall c \in Cuts : \forall l \in Lines : c \cap l \neq \emptyset$. □

The axiom of K-density formalizes our intuitive idea that every cut meets every line: If we take an arbitrary snapshot of our system, every line should appear in this snapshot. Certainly it is impossible for propagating signals to jump over some particular cut. K-density requires the existence of certain elements in the intersection of a line and a cut without necessarily leading to conventional density.

**Definition D2** [Proximity-Relation] $x\ P\ y :\Leftrightarrow \underline{li}_X[x] \subset \underline{li}_X[y]$. □

$\underline{li}_X[x] \subseteq \underline{li}_X[y]$ indicates that every element that has a causal influence on $x$ also affects $y$. This means the causal influence of those elements in $li[x]$ is conveyed to $y$ via $x$. As we have even the stronger condition $\underline{li}_X[x] \subset \underline{li}_X[y]$ there are elements affecting $y$ which do not have any influence on $x$. If these elements affect $y$, this influence must be carried to $y$ via some element different from $x$. Therefore $y$ can be conceived as the center of an interaction directly involving all elements of $P^{-1}[y]$, which are the signals participating in this interaction. Viewing an interaction between signals as a local change of states, the center $y$ will be called an *event* and $x$ is a *local state* that *is changed by* $y$.

**Axiom A9** [No Changes of Changes] $P^2 = \emptyset$. □

With this axiom we require that a local state cannot be an event and vice versa. If $x$ is changed by $y$ ($x\ P\ y$) then $y$ is an event and cannot be changed by some $z$ ($y\ P\ z$ is impossible). This axiom is fundamental in our theory, since together with the other axioms it induces a partition of $X$ into local states and events. It will turn out that every element is either located in the range of $P$ (if it is an event) or in the domain of $P$ (if it is a signal). Note, furthermore, that this is the first axiom, which introduces an asymmetry between causality and concurrency indicating a fundamental difference between time and space.

**Definition D3** [Immediate Neighborhood] $im := P \cup P^{-1}$. □

Typically the notion of change is introduced with the help of a local neighborhood. Here exactly the opposite way has been choosen, since the concept of change arises naturally (and seems to be more fundamental) as it was introduced above. The immediate (temporal) neighborhood $im[x]$ of some event $X$ contains all local states which are changed by this event. The immediate neighborhood $im[x]$ of a local state $x$ contains exactly those events which change $x$.

**Definition D4** [Details and Detail Neighborhood]

a) $x\ D\ y :\Leftrightarrow \underline{co}_X[x] \subset \underline{co}_X[y]$;
b) $dn := D \cup D^{-1}$.

□

Formally similar to $P$ it is possible to introduce a relation $D$, where $x\ D\ y$ holds, iff every element which is concurrent to $x$ is also concurrent to $y$, and there are elements concurrent to $y$ but not concurrent to $x$. We will see later that $x\ D\ y$ may be interpreted as "$x$ is a detail of $y$". The physical significance of details is not yet fully understood, so we do not require any axioms here concerning $D$.

**Axiom A10** [Coherence on Lines] $\forall l \in Lines(CS) : (im|l)^*_l = l \times l$. □

For each world-line it will be required that between every pair of elements on this line there is a finite $im$-chain which is completely covered by that line. Thus taking two elements from some line, the effect of one element on the other occurs within a finite number of steps (changes). This indicates a further manifestation of the finiteness already mentioned in connection with the (first) axiom of coherence. Again infinite world-lines are not excluded by this axiom.

The following two axioms are concerned with the postulate that each of the elements of our system should be capable to infer the arrow of time by local rules without ambiguity. Once the system evolves in some direction of time, this direction is never changed (in analogy to the conservation of momentum) and there is no part of the system which might stop the evolution. The arrow of time which can be conceived as a relation $F$, which orients the relation of immediate (temporal) neighborhood in some consistent manner (all we need here is: $F \cup F^{-1} = im$, $F \cap F^{-1} = \emptyset$ and $F^2 \subseteq li$). A complete definition of "consistent orientation" is not necessary here, as it is possible to motivate the following two axioms by pure symmetry arguments.

**Axiom A11** [Local Transitivity of Concurrency] $\forall x \in X : (co|im[x])^2 \subseteq \underline{co}_X|im[x]$. □

From the axioms of symmetry, disjointness, completeness and coherence it will be shown in the subsequent section that concurrency cannot be a transitive relation. What we postulate with this axiom is a local transitivity of concurrency within the immediate neighborhood $im[x]$ of each element $x$. Local transitivity can be justified as follows: Assume there are three elements $a, b, c$ in $im[x]$ violating local transitivity, e.g. $a\ co\ b$, $b\ co\ c$ and $a\ li\ c$. Then, if we choose an arrow of time $F$ requiring that $a$ occurs before $x$ and $x$ occurs before $c$, the temporal order between $b$ and $x$ is ambiguous, since we have a local symmetry ($a$ and $c$ could be exchanged). It is this kind of ambiguity which is avoided by this axiom. Furthermore every artificial ordering of $x$ and $b$ would violate the initial assumption of concurrency between either $b$ and $a$ or $b$ and $c$. Another justification for the local transitivity of concurrency is the transitivity of simultaneity, which is the limit of concurrency for small distances (here the smallest conceivable distance is given by the $im$-relation).

**Axiom A12** [Local Orientability] $\forall x \in X : (li|im[x])^2 \subseteq \underline{co}_X|im[x]$. □

Assume that for some element $x$ there are three elements $a, b, c$ in $im[x]$ and the axiom does not hold. Then these three elements might be causally dependent of each other, i.e. $a\ li\ b$, $b\ li\ c$ and $a\ li\ c$. From the (local) view of element $x$ the arrow of time concerning $a, b, c$ is not determined: If we require that $a$ occurs before $x$ and $x$ occurs before $c$, for reasons of symmetry (we could exchange $c$ and $a$ or $c$ and $b$) the temporal order between $b$ and $x$ is ambiguous. Loosely speaking, $x$ cannot determine the arrow of time by some local rule. Every local orientation would be arbitrary and this is what we exclude by this axiom.

**Axiom A13** [Local Extendability] $\forall x \in X : id_X|im[x] \subseteq (li|im[x])^2$. □

This axiom postulates that every element located within the immediate neighborhood of some $x$ has at least one temporal predecessor or successor, which can be locally determined.

Given $a$ in $im[x]$ the set of possible successors or predecessors is simply $(li|im[x])[a]$, which is not empty by this axiom.

This completes our list of axioms, which will serve as a starting point for the following investigation. The major changes to the original axioms of concurrency theory are the following: The original axiom of $im$-coherence $im_X^* = X \times X$ has been replaced by the stronger version of coherence on lines, to avoid concurrency structures, where between two elements on a line there is an infinite number of further elements. The axiom of finite concurrency neighborhood, $\forall x \in X : co[x]$ is finite, which was not present in the original formulation, has been chosen as an additional axiom. As the later property will turn out to serve as a sufficient condition for the cone intersection property, it is not necessary to ensure this property by a further axiom, which would be difficult anyway, as the original formulation of the cone intersection property is based on the concept of partial orders. As one section is devoted to this subject, we will get more into detail there.

## Examples

In Fig. 1.a we see the smallest known model satisfying these axioms. As $|\ li\ |$ is large only $co$ is shown and $li = \overline{co}_X - id_X$ is assumed. $P$ is also shown, although it can be derived from $li$ as it was defined above. Fig. 1.b shows a regular, infinite concurrency structure, which could be imagined as the unfolding of the previous one. Fig. 2.a gives a larger model with its infinite unfolding, which is also a concurrency structure, in Fig. 2.b. The excact relations of the finite models are given in the appendix and the axioms have been verified automatically.

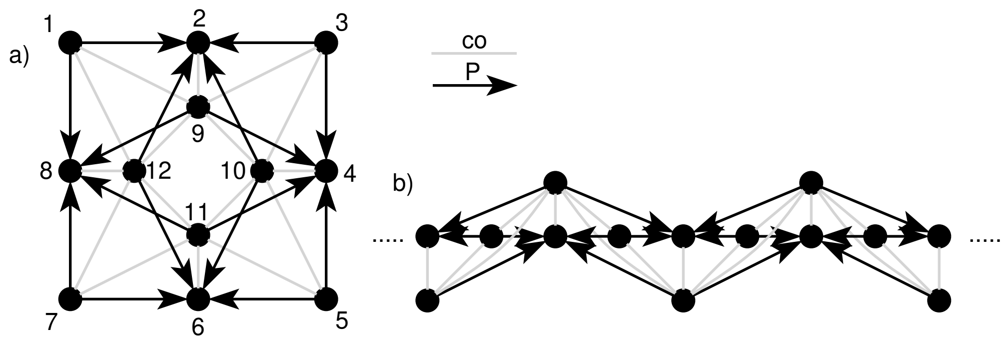


Figure 1: The smallest known concurrency structure and its infinite unfolding.

□ S1

# 5 Properties of Concurrency Structures

In the following sections first some basic properties of concurrency structures are derived. Later the relation to nets and to elementary net systems will be shown, which constitute the bridge to the upper levels of General Net Theory.

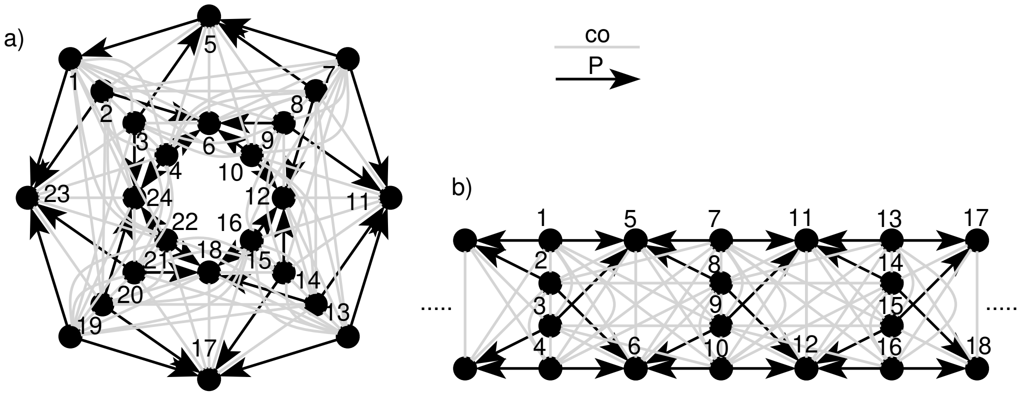


Figure 2: Another finite concurrency structure and its infinite unfolding.

We start with a summary of definitions already introduced above:

**Definition D5** Let $CS = (X, li, co) \wedge x, y \in X \wedge li, co \subseteq X \times X$.

a) $Lines(CS) := Kens(\underline{li}_X)$;
b) $Cuts(CS) := Kens(\underline{co}_X)$;
c) $x\ P_{CS}\ y :\Leftrightarrow x\ P\ y :\Leftrightarrow \underline{li}_X[x] \subset \underline{li}_X[y]$;
d) $x\ D_{CS}\ y :\Leftrightarrow x\ D\ y :\Leftrightarrow \underline{co}_X[x] \subset \underline{co}_X[y]$;
e) $im_{CS} := im := P \cup P^{-1}$;
f) $dn_{CS} := dn := D \cup D^{-1}$.

□

The axioms of concurrency theory we have already discussed are collected in the following definition of a *concurrency structure*:

**Definition D6** $cs(CS) :\Leftrightarrow$

a) $\mid X \mid > 1$;
b) $co \cup li \cup id_X = X \times X$;
c) $co \cap li = li \cap id_X = co \cap id_X = \emptyset$;
d) $co^{-1} = co$;
e) $\tilde{co}_X = \tilde{li}_X$;
f) $co^*_X = li^*_X$;
g) $\forall x \in X : co[x]$ is finite;
h) $\forall c \in Cuts(CS) : \forall l \in Lines(CS) : c \cap l \neq \emptyset$;
i) $P^2 = \emptyset$;
j) $\forall l \in Lines(CS) : (im|l)^*_l = l \times l$;
k) $\forall x \in X : (co|im[x])^2 \subseteq \underline{co}_X|im[x]$;
l) $\forall x \in X : (li|im[x])^2 \subseteq \underline{co}_X|im[x]$;
m) $\forall x \in X : id_X|im[x] \subseteq (li|im[x])^2$.

□

For everything that follows we assume that $CS$ is a concurrency structure satisfying all these axioms.

**Scope S2** Let $CS = (X, li, co) \wedge cs(CS)$.

Concurrency was postulated to be symmetric. It trivially follows that causality must be symmetric too, such that the formal symmetry between $co$ and $li$ (which indicates a symmetry between time and space up to a certain degree) is not violated by this axiom.

**Proposition P1** $li^{-1} = li$. $\square$

**Proof** Assume there are $x, y \in X$ such that $x\ li\ y$ and $\neg y\ li\ x$. Then D6b and D6c require that $y\ co\ x$ and $\neg x\ co\ y$. This is impossible with $co = co^{-1}$. $\square$

The following remark gives a more compact notation of $\tilde{R}_X$, that was used in D6e to postulate the extensionality principle of concurrency theory, which is derived in the next proposition.

**Remark R1** Let $R \subseteq X \times X \wedge R$ be symmetric.
$\tilde{R}_X = \overline{((R \odot \overline{R}_X) \cup (\overline{R}_X \odot R))}_X$. $\square$

**Proof** By definition $a\ \tilde{R}_X\ b \Leftrightarrow \forall x \in X : (a\ \underline{R}_X\ x \Leftrightarrow b\ \underline{R}_X\ x)$. $a\ \tilde{R}_X\ b$ is equivalent to $\neg \exists x \in X : (a\ \underline{R}_X\ x \wedge \neg x\ \underline{R}_X\ b) \vee (\neg a\ \underline{R}_X\ x \wedge x \underline{R}_X\ b)$ which is itself equivalent to $\neg((a\ R \odot \overline{R}_X b) \vee (a\ \overline{R}_X \odot R\ b))$. $\square$

As a direct consequence of D6e we find that it is actually true that two elements are identical if and only if their relations of concurrency and causality to all other elements are identical. Later on we will see that this extensionality principle has a natural counterpart on the level of nets.

**Proposition P2** $\tilde{co}_X = id_X \wedge \tilde{li}_X = id_X$. $\square$

**Proof** It is clear that $\tilde{co}_X \subseteq \underline{co}_X$ and $\tilde{li}_X \subseteq \underline{li}_X$. As we know furthermore $\underline{co}_X \cap \underline{li}_X = id_X$ it is necessary that $\tilde{co}_X \cap \tilde{li}_X \subseteq id_X$. Together with D6e we get $\tilde{co}_X \subseteq id_X$ and $\tilde{li}_X \subseteq id_X$. And finally it is evident that $id_X \subseteq \tilde{co}_X$ and $id_X \subseteq \tilde{li}_X$. $\square$

From axiom D6f we can derive that for each arbitrary pair of elements there is a finite *co*-chain as well as a finite *li*-chain. Note that this is actually a very weak finiteness, since it does not imply that all chains between two elements are finite.

**Proposition P3** $co^*_X = X \times X \wedge li^*_X = X \times X$. $\square$

**Proof** We prove: $X \times X \subseteq co^*_X$. Assume there are $x, y \in X$ with $\neg x\ co^*_X\ y$. This implies $x\ li\ y$ (by $co \cup li \cup id_x = X \times X$). But this contradicts $co^*_X = li^*_X$. The proof remains valid if we exchange $co$ and $li$. $\square$

The last proposition reveals an important property of concurrency structures which is the non-transitivity of concurrency and causality (more precisely $\underline{co}_X$ and $\underline{li}_X$). This point is an essential difference to approaches postulating transitivity of concurrency !

**Corollary C1**

a) $co^*_X \neq \underline{co}_X$;
b) $li^*_X \neq \underline{li}_X$.

$\square$

**Proof** $\underline{co}_X = X \times X$ implies $li = \emptyset$ (D6b, D6c) which contradicts $li^*_X = X \times X$ if $\mid X \mid > 1$ (D6a). The proof for $li$ is analogous. $\square$

## 5.1 Partial Orders

Given an arbitrary partial order $(X, \leq)$ we can derive a causality relation $li = (\leq \cup \leq^{-1}) - id_X$ and a concurrency relation $co = \overline{li}_X - id_X$ (the relation of disorder). In this way we can separate a special class of concurrency structures, which can be represented by partial orders. This is a proper subclass as concurrency structures may by cyclic in general, such that they cannot be covered completely by the formalism of partial orders. Nevertheless is is convenient to introduce the following definition:

**Definition D7** Let $poset(X, \leq)$.
$cs(X, \leq) :\Leftrightarrow cs(X, li, co)$ where
$li = (\leq \cup \leq^{-1}) - id_X$ and $co = \overline{li}_X - id_X$. □

It is clear that the same concurrency structure in terms of $co$ and $li$ is given by a partial order and its converse.

**Remark R2** Let $poset(X, \leq)$.
$cs(X, \leq) \Leftrightarrow cs(X, \leq^{-1})$. □

Yet it is not said whether the orientation of the partial order is somehow related to the arrow of time (which is not yet defined). We will discuss this point in a more general setting when we deal with the orientation of concurrency structures. Choosing partial orders as the fundamental structure is a conventional approach to process theory (*Best und Fernández 1988*). Here it is intentionally tried to deal with concurrency theory in its general form, although several difficulties arise, as the proof techniques known from partial orders cannot be simply applied (in particular one cannot exploit the nice property of transitivity).

## 5.2 Cuts and Lines

In Minkowski-Space we can define space-like surfaces and world-lines with the help of light cones, such that a world-line is always contained in some cone, and a space-like surface has to be contained in the complement of some cone. In concurrency theory similar notions are desired, but the definition is not immediately obvious since, as we have seen in P3, concurrency and causality cannot be transitive. In particular they are no equivalence relations such that the concept of equivalence classes of $co$ and $li$ have to be replaced by a more general idea, which can be applied to relations which are only partially transitive (transitivity with respect to a certain subset of elements). The solution is not difficult: Cliques of a relation denote exactly those subsets of elements which are transitive with respect to concurrency or causality. Kens are maximal cliques, that is, they cannot be extended by adding further elements.

Lines are defined to be Kens of causality, and cuts are Kens of concurrency (more precisely, $\underline{co}_X$ and $\underline{li}_X$). In our standard interpretation a cut corresponds to the complete, spatially distributed state. A line is a set of elements (signals and events) similar to a world-line but note that lines as well as cuts are unordered sets.

In contrast to world-lines and space-like cuts in Minkowski-Space, where between every two points we can find a further one (density), we required the axiom of K-density, which guarantees that our structure is "dense"-enough to ensure that every line intersects with every cut. We may think of K-density as the actual purpose of density, although we should are aware of the fact that in Minkowski-Space density fails to satisfy this purpose, as it is not K-dense (there is a world-line and a cut which do not meet each other).

From the assumption that the intersection of a cut and a line is not empty we can even infer that they must meet in exactly one element as the following proposition shows.

**Proposition P4** $\forall c \in Cuts(CS) : \forall l \in Lines(CS) : \mid c \cap l \mid = 1.$ □

**Proof** Assume there are $c, l$ with $x, y \in c \cap l$ and $x \neq y$. Then $x$ *co* $y$ and $x$ *li* $y$ which contradicts D6c. Thus $\forall c \in Cuts(CS) : \forall l \in Lines(CS) : \mid c \cap l \mid \leq 1$ and together with D6h we get the proposition. □

That each clique can be extended to a ken is clear for finite structures, but we have to apply Zorn's Lemma to prove this in general.

**Proposition P5** Let $C$ be a clique of $R$. Then $\exists K \in Kens(R) : C \subseteq K$. □

**Proof** Let $CL = \{C' : C'$ is a clique of $R$ and $C \subseteq C'\}$ and notice that $(CL, \subseteq)$ is a poset. Choose an arbitrary chain $cl \subseteq CL$ such that $(cl, \subseteq)$ is a total poset. Define $s_{cl} \in Sup(\subseteq, cl)$ which is $s_{cl} = (\bigcup cl) \in CL$. Since $cl$ was arbitrary every chain $cl$ has a supremum within $CL$. Applying Zorn's Lemma we find $\exists K : K \in Max(\subseteq, CL)$ which shows that $K$ is a ken of $R$ with $C \subseteq K$. □

As a simple but useful corollary we find that every element can be extended to a line as well as to a cut. Furthermore every pair of elements which are concurrent or causally dependent is part of some cut or line, respectively.

**Corollary C2**

a) $\forall x \in X : \exists l \in Lines(CS) : x \in l$;
b) $\forall x \in X : \exists c \in Cuts(CS) : x \in c$;
c) $\forall x, y : x \ li \ y \Rightarrow \exists l \in Lines(CS) : x, y \in l$;
d) $\forall x, y : x \ co \ y \Rightarrow \exists c \in Cuts(CS) : x, y \in c$.

□

As every element of $X$ is contained is some line and some cut, the following corollary is immediate, which states that the whole structure is covered by lines as well as cuts.

**Corollary C3**

a) $X = \bigcup Lines(CS)$;
b) $X = \bigcup Cuts(CS)$.

□

From the axiom of finite concurrency neighborhood D6g it immediately follows that all cuts are finite. That infinite lines are possible, on the other hand, is illustrated by the example in Fig. 1.b. So the causality neighborhood $li[x]$ of some element $x$ is not necessarily finite, so we recognize a further asymmetry between concurrency and causality in our formulation of the theory.

**Remark R3** $\forall c \in Cuts(CS) : c$ is finite. □

**Proof** A direct consequence of D6g. □

K-density as it was already required by D6h can be defined independently of concurrency structures for any symmetric and reflexive relation $R$ on a set $X$. In contrast to K-density which requires a global view of $R$ (we have to determine kens) N-density is a local form of density, as we have to verify the property given below only within the neighborhood of each tuple $(a,b,c,d)$. The name N-density comes from the fact that we have to look at those elements $(a,b,c,d)$ that resemble the shape of the letter N with respect to $co$ as well as $li$ (this is the left-hand-side of the implication below). Loosely speaking, $N$-density postulates the existence of an element in the intersection of the N of $li$ with the N of $co$ (the right-hand-side of the implication).

**Definition D8** Let $R \subseteq X \times X \wedge R \cap id_X = \emptyset \wedge R^{-1} = R \wedge S = \overline{R}_X - id_X$.

a) $KDense(X,R) :\Leftrightarrow \forall r \in Kens(\underline{R}_X) : \forall s \in Kens(\underline{S}_X) : r \cap s \neq \emptyset$;
b) $NDense(X,R) :\Leftrightarrow (\forall a,b,c,d \in X : a\ R\ c \wedge b\ R\ d \wedge c\ R\ d \wedge a\ S\ b \wedge a\ S\ d \wedge b\ S\ c \Rightarrow \exists e \in X : c\ R\ e \wedge d\ R\ e \wedge a\ S\ e \wedge b\ S\ e)$.

□

The following proposition shows that K-density implies N-density. As there are relations which are N-dense but not K-dense, N-density could be characterized as the local aspect of K-density.

**Proposition P6** $KDense(X,R) \Rightarrow NDense(X,R)$. □

**Proof** Let $KDense(X,R)$ **(1)** and $a,b,c,d \in X$ with $a\ R\ c \wedge b\ R\ d \wedge c\ R\ d \wedge a\ S\ b \wedge a\ S\ d \wedge b\ S\ c$. Certainly we have some $l \in Kens(\underline{S}_X)$ with $a,b \in l$ and some $c \in Kens(\underline{R}_X)$ with $c,d \in c$. By 1 the must be some element $e$ with $e \in c \cap l$. By definition of $Kens$ it is necessary that $e\ R\ d \wedge e\ R\ c$ and $e\ S\ a \wedge e\ S\ b$. And this is exactly the $e$ required by our proposition. □

The converse is not true, even not for finite relations $R$ as we see in Fig. 3 which gives a structure which is N-dense but not K-dense.

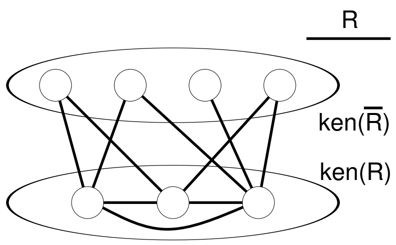


Figure 3: N-dense but not K-dense

As a direct consequence of D6h we have K-density and N-density with respect to concurrency as well as causality.

**Remark R4**

a) $KDense(X,co) \wedge KDense(X,li)$;
b) $NDense(X,co) \wedge NDense(X,li)$.

□

The following example shows that the axiom of K-density is independent of the other concurrency axioms (even for the finite case): Fig. 4 gives a slight modification of the concurrency structure, that was already shown in Fig. 1. It is not K-dense (as it is not N-dense: $a$,$b$,$c$,$d$ constitute the N) but satisfies all other axioms. Furthermore this example shows that results found for (occurrence) posets in *Best und Fernández 1988* (e.g. every occurrence poset is N-dense) cannot be easily transferred to concurrency structures (which are no posets).

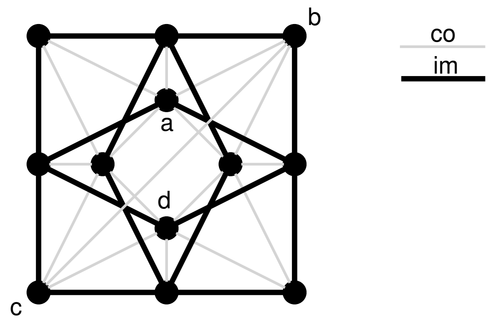


Figure 4: Satisfies all axioms except for K-density

That K-density cannot be derived from N-density (at least for infinite structures) if we satisfy all Concurrency Axioms except for D6h is shown by a simple model, which is a two dimensional infinite grid with respect to the relation $im$. The formal definition is:

$X = \{(x,y) : (x,y) \in \mathbb{Z} \times \mathbb{Z} \wedge \neg(odd(x) \wedge odd(y))\}$;
$(x_0,y_0)\ \underline{co}_X\ (x_1,y_1) :\Leftrightarrow ((x_0 \geq x_1) \wedge (y_0 \geq y_1)) \vee ((x_1 \geq x_0) \wedge (y_1 \geq y_0))$;
$co := \underline{co}_X - id_X$;
$li := X \times X - co - id_X$.

$im$ and some part of $co$ and $li$ is shown in Fig. 5. $(X, li, co)$ is N-dense but not K-dense: The line $l = \{(x,0) : x \in \mathbb{Z}\}$ does not intersect with the cut $c = \{(x,1) : x \in \mathbb{Z} \wedge even(x)\}$.

A finite example of the last kind in not known. Hence it is an open question, if for finite structures we could replace the axiom of K-density by N-density without changing the class of models.

K-density plays a crucial role when building the bridge to elementary net systems. Furthermore K-density is one of the necessary ingredients for D-continuity. In that context it is used to ensure gap-freeness. The undesired jumps are excluded by additional requirements. For further details concerning K-density of partial orders it is refered to *Fernández und Thiagarajan 1985*, *Plünnecke 1985* and *Best und Fernández 1988*. General K-density of concurrency structures, which are not partial orders, has not been investigated much.

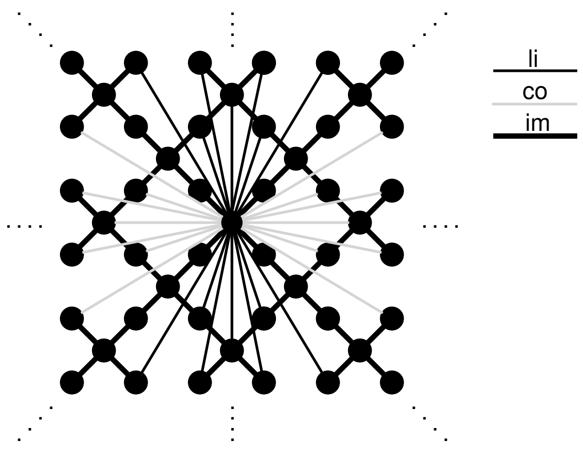


Figure 5: N-dense but not K-dense

### 5.3 Local States and Events

According to the principle of locality in one step of system evolution an element influences or is influenced only by elements, which are located in the immediate neighborhood. In fact we have to deal with the immediate temporal neighborhood here, as it is the relation of causality from which the influence emerges. But how is it possible to derive the immediate neighborhood from a causality relation, which apparently does not reveal any information about the distance between elements? It turns out that the axiom of extensionality (D6e) plays a crucial role as it guarantees that for two distinct elements $x, y \in X$ we are always sure that $\underline{li}_X[x] \neq \underline{li}_X[y]$, which means that causality cones of two elements are always different. So it is a natural assumption that the distance between two elements is completely expressed by the relation of causality. Astonishingly, it is appropriate to define two elements to be immediate neighbors if and only if their causality cones are comparable. Intuitively (if we imagine the light cones in Minkowski-space) this cannot be satisfied, if two elements are far away from each other. And this is exactly the way we have defined the symmetric relation $im := P \cup P^{-1}$ of immediate neighborhood on the basis of the proximity relation $P$ ($x\ P\ y :\Leftrightarrow \underline{li}_X[x] \subset \underline{li}_X[y]$). In this sense $x\ P\ y$ means that the causality cone of $x$ is contained (as a proper subset) in the causality cone of $y$.

It could be argued that this is a very strict requirement for immediate neighborhood, but there seems to be no alternative. Fortunately it follows from D6j that the neighborhood cannot be empty (C4). Indeed, it can be proved from the coherence of lines (D6j) that a concurrency structure is coherent in terms of $im$, such that between every two elements we can find a finite $im$-chain as the following proposition shows.

**Proposition P7** $im_X^* = X \times X$. □

**Proof** Certainly we have $\forall l \in Lines(CS) : (im|l)_l^* = l \times l$ by D6j and $li_X^* = X \times X$ by P3. So between every two elements $x, y \in X$ we can find a finite $li$-chain ($x = x_0, x_1, \ldots, x_n = y$). For every pair $x_i, x_{i+1}$ there is a line $l \in Lines(CS)$ with $x_i, x_{i+1} \in l$ and by D6j a finite $im$-chain ($x_i = z_{i,0}, z_{i,1}, \ldots, z_n = x_{i+1}$) which proves that $x_i\ im_X^*\ x_{i+1}$ and $x\ im_X^*\ y$.

□

In the original proposals for concurrency axioms it was exactly this proposition P7, which was required as an axiom instead of D6j. In *Best und Merceron 1985* it was recognized that D-continuitiy does not follow with that choice of axioms. One reason is that D6j does not generally hold within that system, so we have chosen it as an axiom.

**Corollary C4** $\forall x \in X : im[x] \neq \emptyset$. □

As we have seen, the derivation of the immediate neighborhood from causality naturally leads to $P$ the proximity relation as a by-product. In the following we try to motivate our initial interpretation of $x\ P\ y$ as "$x$ is changed by $y$" from a different point of view. First we observe that $P$ is an asymmetric and irreflexive relation.

**Proposition P8**

a) $P \cap P^{-1} = \emptyset$;
b) $P \cap id_X = \emptyset$.

□

**Proof** By D5c. □

Furthermore it is evident that $P \subseteq li$ and $im \subseteq li$, as our intuitive idea of temporal neighborhood suggests.

**Proposition P9** $P \subseteq li$. □

**Proof** Assume $x\ P\ y$. By definition of $P$ we have $\underline{li}_X[x] \subset \underline{li}_X[y]$ and in particular $x \in \underline{li}_X[y]$ which implies $x = y$ or $x\ li\ y$. But $x = y$ is excluded by P8. □

**Corollary C5** $im \subseteq li$. □

As every element has a neighborhood (P7), we know that this element is located in the range or in the domain of $P$. Those elements contained in $Dom(P)$ will be called $S$-elements or local states, and elements of $Ran(P)$ will be called $T$-elements or events. This will be justified later when it will turn out that depending on the element type ($S$ or $T$) there will be different constraints on the number of immediate neighbors ($im[x]$). Notice that this definition indicates a certain relation to places and transitions in the formalism of nets, and later it will turn out that this is indeed the case.

**Definition D9**

a) $S_{CS} := S := Dom(P)$;
b) $T_{CS} := T := Ran(P)$.

□

**Remark R5** $S \cup T = X$. □

**Proof** From $im_X^* = X \times X$ (P7) we get $(P \cup P^{-1})_X^* = X \times X$. Then $\forall x \in X : \exists y \in X : x\ P\ y \vee y\ P\ x$ and $\forall x \in X : x \in Dom(P) \vee x \in Ran(P)$. □

Each element is either a local state or an event and not both. This is explicitly required by D6i. In this way we have established a partition of $X$ into local states and events. Generally we associate local states with passive entities, which are changed by events as active instances. Notice that this is indeed consistent with our interpretation of $P$.

**Remark R6** $S \cap T = \emptyset$. □

**Proof** By D6i. □

See Fig. 1 and 2 where the partition in $S$-elements and $T$-elements can be easily imagined as $P$ is always directed from $S$ to $T$.

Triangles (i.e. closed chains of size 3) of $im$ are excluded, as the following proposition shows. This can be easily generalized by induction to the statement that closed $im$-cycles of odd length do not occur.

**Proposition P10** $im^2 \subseteq \overline{im}_X$. □

**Proof** Imagine our proposition does not hold. Then we can find an $im$-triangle $x, y, z$ with $x\ im\ y \wedge y\ im\ z \wedge z\ im\ x$. Due to P8 and D5e we have either $x\ P\ y$ or $y\ P\ x$. Proceeding with the former case (the later case is analogous) we need $z\ P\ y$ to ensure that $P^2 = \emptyset$ (D6i). Finally we have to choose $x\ P\ z$ or $z\ P\ y$ yielding a contradiction with $P^2 = \emptyset$ is both cases. □

**Proposition P11** $\forall n \in \mathbb{N} : im^{2n} \subseteq \overline{im}_X$. □

In the next two lemmas some useful relations between $P$, $co$ and $li$ will be derived that will be applied subsequently in several proofs.

The first lemma shows that $x\ P\ y$ holds, iff and only if we have $x\ li\ y$ and there is no $z$, such that $x\ li\ z$ and $z\ co\ y$.

**Lemma L1** $P = li - li \odot co$. □

**Proof**
At first we show $x\ P\ y \Rightarrow x\ li\ y \wedge \neg x\ (li \odot co)\ y$. Assume $x\ P\ y$. $P \subseteq li$ yields $x\ li\ y$ (P9). Assume $x\ (li \odot co)\ y$. Then $\exists z : x\ li\ z \wedge z\ co\ y$ implies $z \in \underline{li}_X[x] \wedge z \notin \underline{li}_X[y]$. Contradiction with $x\ P\ y$ which is equivalent to $\underline{li}_X[x] \subset \underline{li}_X[y]$.
Now it is left to prove $x\ li\ y \wedge \neg x\ li \odot co\ y$ **(1)** $\Rightarrow x\ P\ y$. Assume $\neg x\ P\ y$ which is equivalent to $\neg(\underline{li}_X[x] \subset \underline{li}_X[y])$ **(2)**. Then there are two possibilities: $\neg x\ li\ y$ (which contradicts 1) or $x\ li\ y$, which requires in combination with 2 that $\exists z : z\ \underline{li}_X\ x \wedge \neg z\ \underline{li}_X\ y$. This is equivalent to $\exists z : z\ \underline{li}_X\ x \wedge z\ coy$. For $z = x$ we have $x\ co\ y$, which is not reconcilable with $x\ li\ y$. So we conclude $\exists z : z\ li\ x \wedge z\ co\ y$ or equivalently $x\ li \odot co\ y$. But this is a contradiction with our assumption 1. □

Moreover, if we have $x\ P\ y$ this requires the existence of some $z$, such that $x\ co\ z$ and $z\ li\ y$.

**Lemma L2** $P \subseteq co \odot li$. □

**Proof** By D5c $x\ P\ y$ is equivalent to $\underline{li}_X[x] \subset \underline{li}_X[y]$. This suggests $\exists z : z \notin \underline{li}_X[x] \wedge z \in \underline{li}_X[y]$ implying (by D6c and D6b) that $\exists z : z \in co[x] \wedge z \in li[y]$ ($z = y$ is impossible by P9, which requires $x\ li\ y$). Then we have $\exists z : x\ co\ z\ \wedge z\ li\ y$ leading to $x(co \odot li)y$. □

Concurrency structures are not only $im$-coherent but also coherent with respect to the complement of $im$.

**Proposition P12** $(\overline{im}_X)^*_X = X \times X$. □

**Proof** By P3 we have $co^*_X = X \times X$. From $im \subseteq li$ we find $co \subseteq (X \times X - im)$. □

That for each element we have a non-empty neighborhood can be sharpened to the statement that every element must have at least two neighbors. This is a minimal requirement to prevent world-lines from ending somewhere in time-space without the possibility of temporal continuation. Regrettably we will see that this is no guarantee (Actually, we need ASS1.).

**Proposition P13** $\forall x \in X : \mid im[x] \mid \geq 2$. □

**Proof** In C4 we have seen that $\mid im[x] \mid \geq 1$. To satisfy $id|im[x] \subseteq (li|im[x])^2$ (D6m) there must be at least one further element in $im[x]$ otherwise $(li|im[x])^2 = \emptyset$. □

Finally, although this is only a minor simplification, it has been found that a shorter form of the original axiom A13 is sufficient as it is formulated in A14. So the following axiom has to be taken as an alternative to A13.

**Axiom A14** $\forall x \in X : (li|im[x])^2 \neq \emptyset$. □

This axiom postulates that within the immediate neighborhood there are at least two elements which are causally dependent of each other. From the viewpoint of $x$ this corresponds to the existence of at least two local directions of time (the past and the future). In some sense it is even stronger than A13 as it directly implies that the immediate neighborhood has at least two elements.

The proof of the following proposition, which is the same as A13 shows that this apparently weaker axiom would also be sufficient (although A13 is prefered for its physical evidence). For every element $u \in im[x]$ we can find at least one element $u' \in im[x]$ within the same immediate neighborhood which is influenced by $u$. Viewing $u'$ as the temporal successor or predecessor of $u$ this proposition guarantees the local existence of a temporal direction, where the evolution of the system seen from $x$ might continue.

**Proposition P14** Assume A14. Then $\forall x \in X : id|im[x] \subseteq (li|im[x])^2$. □

**Proof** Choose some arbitrary $x \in X$. It is sufficient to prove the claim that for every $u \in im[x]$ we can find some $u' \in im[x]$ with $u\ li\ u'$. A14 suggests the existence of $y, z \in im[x]$ with $y\ li\ z$. So by D6d the claim is already satisfied, if we choose $y$ and $z$ for $u$ and $u'$ or vice versa. Now assume $u \in im[x]$ is a further element with $u \neq y$ and $u \neq z$. According to D6b we have $y\ li\ u$ or $y\ co\ u$. In the former case the claim holds trivially, if we choose $y$ for $u'$. In that later case we cannot have $u\ co\ z$ as this is incompatible with D6k. So we must have $u\ li\ z$ and choosing $z$ for $u'$ prove the claim. □

## 5.4 Details and Observables

Above the immediate (temporal) neighborhood was introduced as a subset of the causality cone of a particular element. Although there is no evident physical interpretation, we could exploit the symmetry between causality and concurrency, which has been maintained by the axioms D6b–D6h and proceed in a similar way defining concurrency cones $\underline{co}_X[x]$ of elements $x$. The formal counterparts of $P$ and $im$ are $D$ and $dn$, respectively. We have defined $x\ D\ y :\Leftrightarrow \underline{co}_X[x] \subset \underline{co}_X[y]$ and $dn := D \cup D^{-1}$. $x\ D\ y$ means that the concurrency cone of $x$ is completely covered by the concurrency cone of $y$, and everything that is concurrent to $x$ is necessarily concurrent to $y$. So, if we observe $x$ as part of a cut, we are always sure that $y$ must be contained in the same cut (the converse is not necessarily true). This leads to a surprisingly natural interpretation of $x\ D\ y$ as "$x$ is a detail of $y$".

Of course the detail relation $D$ is asymmetric and irreflexive as $P$, and it can be only established between concurrent elements.

**Proposition P15**

a) $D \cap D^{-1} = \emptyset$;
b) $D \cap id_X = \emptyset$.

□

**Proof** By D5d. □

**Proposition P16** $D \subseteq co$. □

**Proof** By D5d. □

Detail neighborhood and immediate neighborhood are always disjunct.

**Corollary C6**

a) $dn \subseteq co$;
b) $im \cap dn = \emptyset$.

□

The $D$-relation of the smallest known concurrency structure is shown in Fig. 6.

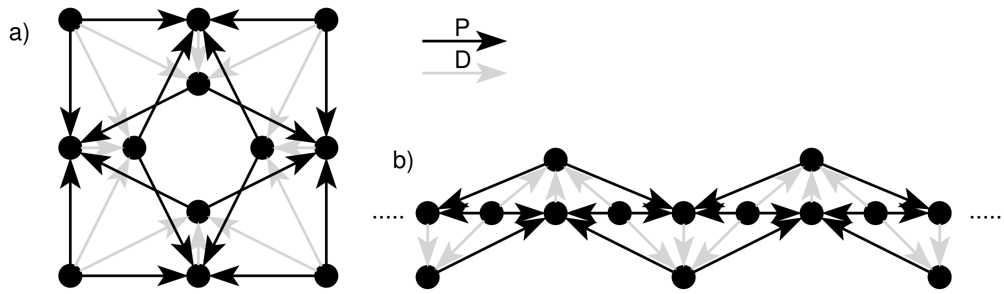


Figure 6: A concurrency structure and its detail-relation

The lemmas L1 and L2 can be translated directly, as they only rely on the lower axioms where symmetry between causality and concurrency is preserved.

**Lemma L3** $D = co - co \odot li$. □

**Proof** Similar to L1. □

**Lemma L4** $D \subseteq li \odot co$. □

**Proof** Similar to L2. □

In Fig. 7 we have another finite concurrency structure $(X, li, co)$ (more precisely given in the appendix) with its detail-relation. It can be easily verified that $D^2 = \emptyset$ and $dn_X^* = X \times X$, which suggests a further symmetry between $D$ and $P$. In fact, this is true for some concurrency structures, but it does not hold in general. A counterexample with $dn_X^* \neq X \times X$ is already given by Fig. 2, where the detail relation is empty. That $D^2 \neq \emptyset$ may occur is proven by the example in Fig. 7.

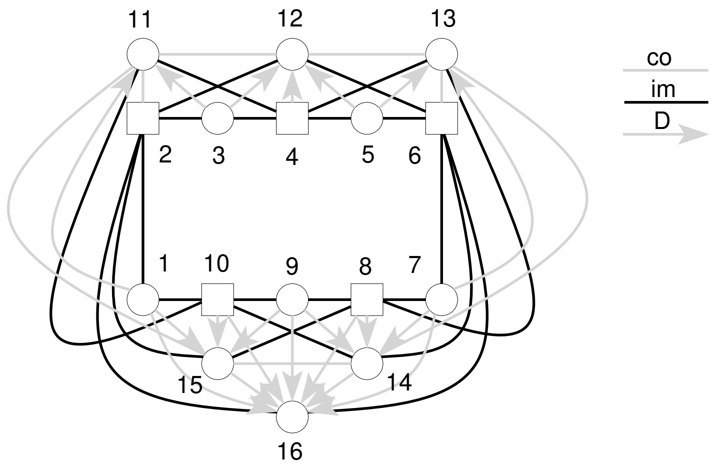


Figure 7: Details of Details

As a consequence, the coherence and non-emptiness of detail neighborhood (the counterparts of P7 and C4 cannot be derived, because an axiom similar to D6j was not required for cuts. For sake of completeness, it should be mentioned that these two properties (detail axioms) were required in *Petri 1987* (for their mathematical beauty but not for physical reasons) but have been given up, probably because it turned out that they are too restrictive.

What is demonstrated by Fig. 7 is that the detail-relation establishes a hierarchy, which can be formally described by the fact that $(X, D)$ is a strict partial order. Indeed, this was already true for $P$ but this order is degenerate, as it is a collection of pairs which are not connected with each other. Recent ideas of Petri indicate that allowing arbitrary strict posets of $P$ leads to a consequent generalization of nets incorporating an interesting concept of dimensionality into net theory. The significance of similar ideas with respect to $D$ is not yet clear.

**Proposition P17**

a) $(X, D)$ is a strict poset;
b) $(X, P)$ is a strict poset.

□

**Proof** This strict poset is inherited from the poset $(\mathcal{P}(\mathcal{X}),\subset)$ in the definitions of $D$ and $P$. □

The interpretation of $D$ already designates all elements of $Dom(D)$ as details. Lead by the notion of observables in quantum mechanics, where due to Heisenberg's uncertainty relation not every detail (of a systems state) can be observed by one measurement, we might try the following (speculating) definition:

**Definition D10**

a) $Details(CS) := Dom(D)$;
b) $Observables(CS) := Ran(D)$.

□

Proceeding in this way we find the intuitively expected property of observables: Every observable is a local state and not an event. But as we have seen in the examples above, not every local state must be necessarily observable, and in general not every event is a detail.

**Proposition P18** $Observables(CS) \subseteq S$. □

**Proof** Assume $x\ D\ t$ with $t \in T$. By D5d there is some $z$ with $x\ li\ z \wedge z\ co\ t$. Since $t \in T$ there must also be a $b$ with $b\ P\ t$. Now what is the relation between $x$ and $b$? According to D5d is cannot be $x\ co\ b$. But by L1 it cannot be $x\ li\ y$. Altogether this leads to contradiction with D6b. □

## 5.5 Immediate Neighborhood

In this section we deal with the effect of axioms D6k, D6l and D6m upon the immediate neighborhood of local states and events. Remember that the purpose of these axioms was that within the space-time of locally interacting elements it should be possible for each element with its restricted view to infer in which temporal direction it has to drive the system's evolution. As boundary conditions it was required that there is no ambiguity in the local time orientation rule the element applies (i.e. there is at most one solution) and there is always a direction in which time evolves (i.e. there is at least one solution). In any case one should exclude a local time reversal, that is, from the local view of an element the direction into which time goes should be different from that where it has come from. The global evolution of a system (which could be imagined as the movement of the global state represented by a cut through time-space) will be uniquely (except for non-determinism due to concurrency) determined by this local evolution rule. The question, if this is sufficient to guarantee the unique existence of a global arrow of time, will be addressed separately later on.

First we recognize that the axiom requiring local transitivity of concurrency (D6k) establishes an equivalence relation in the immediate neighborhood of each element.

**Proposition P19** $\forall x \in X : \underline{co}_X$ is an equivalence on $im[x]$. □

**Proof** That $\underline{co}_X$ is reflexive on $im[x]$ and symmetric is clear. That $\underline{co}_X|im[x]$ is transitive is ensured by $(\underline{co}_X|im[x])^2 \subseteq \underline{co}_X|im[x]$ which follows directly from D6k. □

To exclude local freezing of temporal evolution it is necessary to ensure that causality does not vanish in the scope of the immediate neighborhood.

**Remark R7** $\forall x \in X : li|im[x] \neq \emptyset$. □

**Proof** We know that $im[x] \neq \emptyset$ (C4). By D6m there are $a, b \in im[x]$ with $a\ li\ b \wedge b\ li\ a$ which together with D6c implies $li|im[x] \neq \emptyset$. □

By the help of the equivalence relation introduced above we can decompose the immediate neighborhood of each element $x$ into a number of equivalence classes establishing a partition on $im[x]$. Note that $x$ itself is not contained in $im[x]$, and each of these classes is a clique of $\underline{co}_X$. Furthermore two equivalence classes are not covered by a common cut.

**Remark R8** $\forall g, g' \in im[x]/\underline{co}_X : (g \neq g' \Rightarrow \neg\exists c \in Cuts(CS) : g, g' \subseteq c)$. $\square$

**Proof** Let $g, g' \in im[x]/\underline{co}_X$. Then $\forall x \in g : \forall x' \in g' : x \ co \ x'$ implies $g = g'$. Contradiction. $\square$

Hence we can conceive each of these local equivalence classes as representing (local aspects of) different global states, which arrive/leave the considered element from/in different directions of time. As we have confined ourselves to models without branching time (by excluding alternatives on this level), we should expect that only two equivalence classes exist (corresponding to temporal predecessor and successor states), what is easily proved by the following proposition.

**Proposition P20** $\forall x \in X : \mid im[x]/\underline{co}_X \mid = 2$. $\square$

**Proof** D6m ensures that there are $a, b \in im[x]$ with $a \ li \ b$. This implies $\mid im[x]/\underline{co}_X \mid \geq 2$. $(li|im[x])^2 \subseteq \underline{co}_X|im[x]$ (D6l) shows that $\mid im[x]/\underline{co}_X \mid \leq 2$. $\square$

Applying D6l to local states leads to the the fact that each state has exactly two neighbors. As we know that neighbors of states are always events, we could denote one of them as the sender of a signal and the other one as the receiver.

**Proposition P21** Let $s \in S$. Then $\mid im[s] \mid = 2$. $\square$

**Proof** P13 already shows $\mid im[s] \mid \geq 2$. So there are $x, y \in im[s]$ with $x \neq y$. Now assume there is an additional $z \in im[s]$ with $z \neq x$ and $z \neq y$. $s \in S$ implies $s \ P \ x \wedge s \ P \ y \wedge s \ P \ z$. Then by L1 we must have $x \ li \ z$, $y \ li \ z$ and $x \ li \ y$. Using $(li|im[s])^2 \subseteq \underline{co}_X|im[s]$ (D6l) we derive a contradiction since we have $x \ li \ z$ and $z \ li \ y$ but neither $x \ co \ y$ nor $x = y$. $\square$

The immediate neighborhood is finite. This corresponds directly to the locality principle known from physics that temporal evolution should be governed by local laws. Notice the parallel to cellular automata, where finiteness of neighborhood is motivated in the same manner. Here neighborhood-finiteness emerges trivially from finiteness of all cuts (although lines may be infinite). A more general approach to concurrency theory might require the following proposition as an axiom instead of cut-finiteness, but this may lead to the problem of reachability of an infinite cut on the level of elementary net systems.

**Proposition P22** $\forall x \in X : im[x]$ is finite. $\square$

**Proof** Assume the contrary: There is an element $x$ with infinite neighborhood $\mid im[x] \mid \notin \mathbb{N}$. Then P20 gives us $im[x]/\underline{co}_X = \{E_1, E_2\}$ such that $E_1$ or $E_2$ must be infinite. Assume $\mid E_1 \mid \notin \mathbb{N}$ then there is an infinite cut $c \in Cuts(CS)$ with $E_1 \subseteq c$. Contradiction with R3. $\square$

**Lemma L5** $X$ is countable. $\square$

**Proof** If $X$ is finite this is clear. So assume $X$ is infinite. Fix an arbitrary element $x$. P22 implies that $N_j := \mid \underline{im}^j_X[x] \mid$ is finite for every $j \in \mathbb{N}$. Therefore for every $j$ we can find a function $d_j : \mathbb{N}[0, \ldots, N_j] \rightarrow \underline{im}^j_X[x]$. Furthermore without loss of generality we assume $i <= j$ implies $\forall k \in \mathbb{N}[0, \ldots, N_i] : d_i(k) = d_j(k)$. Combining all $d_j$ for $j \in \mathbb{N}$ we can establish an enumeration of all elements of $X$, that is, a surjective function $d : \mathbb{N} \rightarrow X$ defined by $d(j) := d_j(j)$. $d$ is well-defined as for every $j$ we have $j \in Dom(d_j)$ because

$j <= N_j$ (remember that due to D6j there is an infinite acyclic $im$-chain containing $x$). According to P7 every $z \in X$ is contained in $im^j[x]$ for some $j \in \mathbb{N}$ such that $d$ is surjective on $X$. □

Of course as it is already suggested by D6j the elements on a line are also countable:

**Corollary C7** Let $l \in Lines(CS)$.
Then $l$ is countable. □

### 5.6 The Structure of Lines

So far we have seen lines as unordered sets of elements. With D6j it is postulated that every two elements on a line are connected via a finite $im$-chain, which is completely covered by that particular line. Due to the properties of the proximity relation $P$ (which is necessary for $im$) these chains consist of local states and events appearing in an alternating fashion. So on a line between (to be defined in terms of $im$) two events there is always a state and vice versa.

Considering the intersection of the immediate neighborhood of an element with a line containing this element gives us a natural vehicle to classify elements with respect to that line into four categories:

**Definition D11** Let $l \in Lines(CS)$.

a) $IsolatedPoints_{CS}(l) := \{x : x \in l \wedge |\ (im|l)[x]\ | = 0\}$;
b) $EndPoints_{CS}(l) := \{x : x \in l \wedge |\ (im|l)[x]\ | = 1\}$;
c) $MidPoints_{CS}(l) := \{x : x \in l \wedge |\ (im|l)[x]\ | = 2\}$;
d) $BranchedPoints_{CS}(l) := \{x : x \in l \wedge |\ (im|l)[x]\ | \geq 3\}$.

□

As our intuitive notion of world-lines suggests, elements on a line are neither isolated nor branched.

**Lemma L6** Let $l \in Lines(CS)$. Then $IsolatedPoints_{CS}(l) = \emptyset$. □

**Proof** Let $x \in l$. If $l \cap im[x] = \emptyset$ then D6j cannot be satisfied. □

**Lemma L7** Let $l \in Lines(CS)$. Then $BranchedPoints_{CS}(l) = \emptyset$. □

**Proof** Let $x \in l$ and assume $|\ l \cap im[x]\ | \geq 3$. Then there are $a, b, c \in im[x]$ with $a\ li\ b \wedge b\ li\ c\ \wedge a\ li\ c$. This directly contradicts $(li|im[x])^2 \subseteq \underline{co}_X$ (D6l). □

Moreover, lines should have no endpoints, as this could lead to partially dead systems. Certainly D6m is a necessary condition to ensure this, but is it really sufficient (in combination with the other axioms)? The following assumption has not been proved. On the other hand, it is believed that it holds, as no counterexample has been constructed. In any case it is physically justified, and, if it turns out that it cannot be proved, we should take this as an axiom, e.g. as a stronger form of D6m. Regrettably we cannot do without this assumption, as it is heavily linked with P24 and P28, which themselves provide a essential connection to the formalism of elementary net systems.

**Assumption ASS1** Let $l \in Lines(CS)$. Then $EndPoints_{CS}(l) = \emptyset$. □

Altogether we come the the conclusion that the elements constituting lines are all midpoints, i.e. they have exactly two neighbors on every line.

**Proposition P23** Let $l \in Lines(CS)$. Then $MidPoints_{CS}(l) = l$. $\square$

Once we have established $x\ P\ y$ we can be always sure to find a $z$ with $z\ co\ x$ and $z\ P\ y$. This is a crucial result following from ASS1.

**Proposition P24** $P \subseteq co \odot P$. $\square$

**Proof** We will show $x\ P\ y \Rightarrow \exists z' : x\ co\ z' \wedge z'\ P\ y$. Assuming $x\ P\ y$ we get $\underline{li}_X[x] \subset \underline{li}_X[y]$ (by D5c) such that $\exists z : z\ li\ y \wedge z\ co\ x$. Then there must be a line $l \in Lines(CS)$ with $z, y \in l$. Applying ASS1 there must be $z', z'' \in l$ with $z'\ im\ y \wedge z''\ im\ y$. It is clear that $x \neq z' \wedge x \neq z''$ since $x \notin l$ and $x\ co\ z \wedge z\ li\ z' \wedge z\ li\ z''$. It is necessary that $z'\ co\ x \vee z''\ co\ y$ otherwise we have a line $l' \in Lines(CS)$ with $x, y, z', z'' \in l'$ with $(im|l')[z] = \{x, z', z''\}$ violating ASS1. D6i requires that $z'\ P\ y$ and $z''\ P\ y$. So we have $x\ co\ z' \wedge z'\ P\ y$ or $x\ co\ z'' \wedge z''\ P\ y$. $\square$

One way to find a proof for ASS1 might involve the previous proposition: It might be possible to prove P24 without using ASS1 and then to derive ASS1 from it.

## 5.7 Propagating Concurrency

It was already mentioned that the global time evolution could be seen as the movement of cuts though time-space. So from a given cut representing the current state we have to derive new cuts, which correspond to predecessor or successor states, and all this has to be done on the basis of local rules, such that only the immediate neighborhood of those elements is involved, which contribute to the evolution at that instant of time.

Concretely, if we have an arbitrary element which is concurrent to an event, we can propagate concurrency in such a way that our element is concurrent to all elements in the immediate neighborhood of that event.

**Proposition P25** Let $t \in T$. Then $x\ co\ t \Rightarrow (\forall y \in im[t] : x\ co\ y)$. $\square$

**Proof** Let $x\ co\ t$ and assume there is some $y \in im[t]$ with $x\ \underline{li}_X\ y$. $x = y$ implies $x\ im\ t$, contradicting $x\ co\ t$. $t \in T$ requires $y\ P\ t$. Then we get a direct contradiction with L1 since we have $y\ li\ t$ and $y\ li \odot co\ t$. $\square$

Interestingly this proposition has a counterpart concerning causality and local states: For all elements that are causally related to a state-element the causality can be extended to its immediate neighborhood.

**Proposition P26** Let $s \in S$. Then $x\ li\ s \Rightarrow (\forall y \in im[s] : x\ \underline{li}_X\ y)$. $\square$

**Proof** Similar to the previous proof let $x\ li\ s$ and assume there is some $y \in im[s]$ with $x\ co\ y$. $s \in S$ implies $s\ P\ y$. $s\ li\ y$ and $s\ li \odot co\ y$ together with L1 leads to $\neg(s\ P\ y)$. A contradiction. $\square$

As a direct consequence, given a local state on a line, that line must necessarily contain its immediate neighborhood.

**Proposition P27** Let $s \in S \wedge l \in Lines(CS) \wedge s \in l$. Then $im[s] \subseteq l$. $\square$

**Proof** Combining $|\, im[s]\, | = 2$ (P7) and $|\, l \cap im[s]\, | = 2$ (P23) we immediately get $im[s] = l \cap im[s]$. □

A more complicated but essential propagation rule for concurrency can be derived from ASS1: Given an event $t$ and an arbitrary element $x$, which is concurrent to all elements in one equivalence class of $t$ (remember the definition of local equivalence classes in section 5.5), concurrency can be propagated in such a way that element $x$ is also concurrent to event $t$.

**Proposition P28** Let $t \in T \wedge E \in im[t]/\underline{co}_X$.
Then $(\forall e \in E : x\ co\ e) \Rightarrow x\ co\ t$. □

**Proof** According to P20 we can write $im[t]/\underline{co}_X = \{E_1, E_2\}$ where $E_1 \cup E_2 = im[t] \wedge E_1 \cap E_2 = \emptyset$. Let $x \in X$ such that $\forall e \in E_1 : x\ co\ e$. Now assume $x\ li\ t$. First observe that there is some line $l \in Lines(CS)$ with $t, x \in l$. D6j additionally requires that a finite $im$-chain $(x = x_0, \ldots, x_n = t)$ is contained completely in $l$ $(\{x_0, \ldots, x_n\} \subseteq l)$. As $x\ co\ e$ holds for all $e \in E_1$ the line $l$ must pass through some $s \in im[t] - E_1$ implying $s \in E_2$. Now observe that $t$ is an endpoint of $l$ in the sense that there is no further element of $im[t] - \{s\}$ contained in that line $(im[t] - \{s\} \cap l = \emptyset)$. This is a direct contradiction with ASS1. □

The restriction that we have to consider events (and their immediate neighborhood) is even not necessary, such that a slightly more general form of this proposition is also possible:

**Remark R9** Let $z \in X \wedge E \in im[z]/\underline{co}_X$.
Then $(\forall e \in E : x\ co\ e) \Rightarrow x\ co\ z$. □

**Proof** Combine P25 (for $z \in T$) and P28 (for $z \in S$). □

Applying these propagation rules it is an easy task to rule out lines that are shorter that four elements. This lower bound cannot be improved as we can see from Fig. 1 where we can identify two lines with exactly four elements.

**Proposition P29** Let $l \in Lines(CS)$. Then $|\, l\, | \geq 4$. □

**Proof** Let $l \in Lines(CS)$ **(1)**. By D6a and P3 we have $|\, l\, | \geq 2$. If $|\, l\, | = 2$ we have $l = \{a, b\}$ and as $a\ im\ b$ we can assume $a \in S$ and $b \in T$. Applying P26 to $a$ with the fact that $im[a] = \{b, c\}$ we find that even $\{a, b, c\}$ is a clique of $\underline{li}_X$. Hence we conclude $|\, l\, | > 2$. Now assume $|\, l\, | = 3$ and $l = \{a, b, c\}$. Without loss of generality we can assume $a\ im\ b$ and $b\ im\ c$ (by D6j). Then by D9 and P8 there are two possibilities, either $a \in S$, $b \in T$, $c \in S$ (case **(2)**) or $a \in T$, $b \in S$ and $c \in T$ (case **(3)**). D6m suggests that there must be an element $d \in im[\{c\}]$ with $d \neq b$ and $b\ li\ d$. $b\ li\ c$ and $c\ li\ d$ is clear by C5. In case 2 we have $c \in S$ and assuming $a\ co\ d$ leads to $a\ co\ c$ (applying P26) which contradicts $a\ li\ c$ In case 3 we have $c \in T$ and assuming $a\ co\ d$ leads to $b\ co\ d$ (by P26), again a contradiction. So in both cases we have $a\ li\ d$ and $\{a, b, c, d\}$ is a clique of $\underline{li}_X$, which is not reconcilable with our initial assumption 1 that $l = \{a, b, c\} \in Lines(CS)$. □

### 5.8 Consistent Orientations and Nets

In this section a link will be established between the formalism of nets and concurrency structures. A major characteristic of nets is the partition of net elements into two sorts: active transitions and passive places. Given a concurrency structure $CS = (X, li, co)$ events

$T_{CS}$ and local states $S_{CS}$ were already introduced constituting a partition of $X$, and we will naturally identify transitions and places with events and local states, respectively. A further feature of nets is the possibility of expressing the symmetric relations of causality and concurrency by means of a single directed flow relation. The term flow relation already indicates that a certain direction of time is chosen, in which the flow (of the distributed state in time-space) is oriented. On the level of concurrency structures, however, we had no privileged arrow of time. So we expect that given a concurrency structure we find different nets with different orientations of the flow relation. Later, when the dynamics of nets is investigated, we will see that certain elementary net systems based on these nets are in some sense equivalent to the underlying concurrency structure.

We start this section with the definition of a consistent orientation, which is a relation $F$ orienting each pair of immediate neighbors $x\ im\ y$ in exactly one direction (that is $F \cup F^{-1} = im$ and $F \cap F^{-1} = \emptyset$) and satisfying some further conditions that will be mentioned immediately. From the viewpoint of some element $x \in X$ we interpret that part of the neighborhood $im[x]$ that is given by $F^{-1}[x]$ as those elements that have a direct effect upon $x$ and $F[x]$ as the elements that are directly affected by $x$. In other words, $F[x]$ and $F^{-1}[x]$ are immediate temporal successors and predecessors, respectively. An additional condition of consistent orientation is given by $F \odot F \subseteq li$. This ensures that the relation of causality holds between an element $y$ that directly affects $x$ ($y\ F\ x$) and an element $z$ that is directly affected by $x$ ($x\ F\ z$). We are sure that in this case $y$ and $z$ must be (indirectly) causally dependent, as there is only one element, namely $x$, located between them. On the other hand, if there are two elements $y, z \in {}^\bullet x_F$ that both affect $x$, they can do this only concurrently and similarly $y, z \in x^\bullet{}_F$ which are directly affected by $x$ must be concurrent. The last two conditions are expressed by $F \odot F^{-1} \subseteq \underline{co}_X$ and $F^{-1} \odot F \subseteq \underline{co}_X$.

**Definition D12** Let $F \subseteq X \times X$ and $Y \subseteq X$.
*$F$ is a consistent orientation on $Y$ in $CS$* $:\Leftrightarrow$

a) $F \cap F^{-1} = \emptyset$;
b) $F \cup F^{-1} = im|Y$;
c) $F \odot F \subseteq li$;
d) $F \odot F^{-1} \subseteq \underline{co}_X$;
e) $F^{-1} \odot F \subseteq \underline{co}_X$.

□

For technical reasons, the definition of consistent orientation will be needed with respect to a subset $Y$ of $X$. This corresponds to a partial consistent orientation of a concurrency structure, which will be useful for the inductive approximation of a (total) consistent orientation.

**Definition D13** *$F$ is a consistent orientation on $CS$* $:\Leftrightarrow$
$F$ is a consistent orientation on $X$ in $CS$. □

An example for a consistent orientation $F$ of the concurrency structures given in Fig. 1 is shown in Fig. 8.

**Remark R10** Let $Z \subseteq Y$.
$F$ is a consistent orientation on $Y \Rightarrow F|Z$ is a consistent orientation on $Z$. □

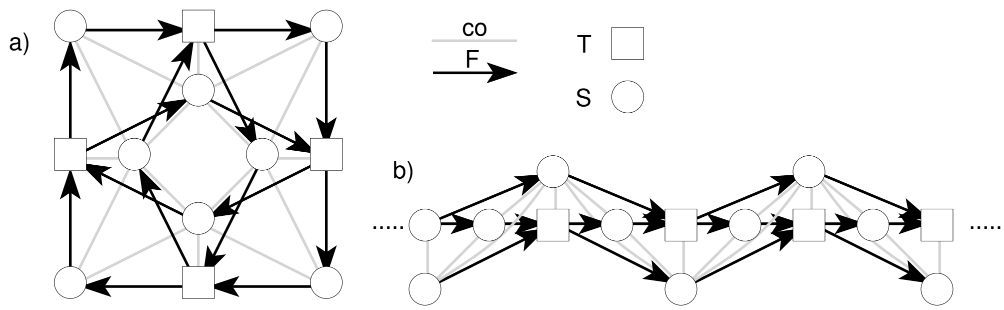


Figure 8: A consistent orientation $F$

An equivalent formulation of the notion of consistent orientation is given by the following remark by simply reversing the implications of the definition D12.

**Remark R11** Let $F \subseteq X \times X$.
$F$ is a consistent orientation on $CS \Leftrightarrow$

a) $\neg(x\ F\ y \wedge x\ F^{-1}\ y)$;
b) $x\ im\ y \Leftrightarrow x\ F\ y \vee x\ F^{-1}y$;
c) $x\ im\ y\ im\ z \wedge x\ li\ z \Rightarrow x\ F\ y\ F\ z \vee x\ F^{-1}\ y\ F^{-1}\ z$;
d) $x\ im\ y\ im\ z \wedge x\ \underline{co}_X\ z \Rightarrow x\ F\ y\ F^{-1}\ z \vee x\ F^{-1}\ y\ F\ z$.

$\square$

So far is has not been proved that such a consistent orientation really exists. It is mainly this question that is to be addressed in the remainder of this section. As mentioned above, a set of nets $Nets(CS)$ will be associated with every concurrency structure $CS$ and the major problem is what we can say about the cardinality of this set.

**Definition D14** $Nets(CS) := \{(S,T,F) : F$ is a consistent orientation on $CS\ \}$. $\square$

Although we do not know if $Nets(CS) \neq \emptyset$, we can easily derive some essential properties of those objects that might be contained in $Nets(CS)$. First some trivial remarks following immediately from the definition:

**Remark R12** Let $(S,T,F) \in Nets(CS) \wedge x \in X$.
Then $\forall y \in {}^\bullet x_F : \forall z \in x^\bullet{}_F : y\ li\ z$. $\square$

**Proof** Immediately from $F^2 \subseteq li$ (D12c). $\square$

**Remark R13** Let $(S,T,F) \in Nets(CS) \wedge x \in X$.

a) $\forall y, z \in {}^\bullet x_F : y\ \underline{co}_X\ z$;
b) $\forall y, z \in x^\bullet{}_F : y\ \underline{co}_X\ z$.

$\square$

**Proof** Immediately from $F \odot F^{-1} \subseteq \underline{co}_X$ and $F^{-1} \odot F \subseteq \underline{co}_X$ (D12). $\square$

**Lemma L8** $im[x] = {}^\bullet x_F \cup x^\bullet{}_F$. $\square$

**Proof** Clear by D12 as $F \cup F^{-1} = im$. $\square$

**Proposition P30** $im[x]/\underline{co}_X = \{{}^\bullet x_F, x^\bullet{}_F\}$. □

**Proof** By R12, R13 and L8. □

In fact, elements of $Nets(CS)$ satisfy the definition of a net, as it was already anticipated above.

**Proposition P31** Let $N \in Nets(CS)$. Then $N$ is a net. □

**Proof** Let $N = (S, T, F) \wedge N \in Nets(CS)$. $S \cap T = \emptyset$ has been found in R6. So all we have to prove is: $F \subseteq (S \times T) \cup (T \times S)$. But this is clear because $P \subseteq (S \times T)$ and $F \subseteq im = P \cup P^{-1}$. □

All nets associated with $CS$ turn out to be connected and pure.

**Proposition P32** Let $N \in Nets(CS)$. Then $N$ is connected. □

**Proof** Let $(S, T, F) \in Nets(CS)$. From $im^*_X = X \times X$ (P7) and $F \cup F^{-1} = im$ we immediately find $(F \cup F^{-1})^*_X = X \times X$. □

**Proposition P33** Let $N \in Nets(CS)$. Then $N$ is pure. □

**Proof** Immediately from $F \cap F^{-1} = \emptyset$ (D12a). □

The elements of these nets are directly affected by at least one element, and they directly affect at least one further element. So the future as well as the past of an element is never empty.

**Proposition P34** Let $(S, T, F) \in Nets(CS)$.
Then $\forall x \in X : |\ {}^\bullet x_F\ | \geq 1 \wedge |\ x^\bullet{}_F\ | \geq 1$. □

**Proof** Let $x \in X$. Assume $x^\bullet{}_F = \emptyset$. C4 and D6m require that there are $y, z \in im[x]$ such that $y\ li\ z$. From our assumption it follows that $y\ F\ x$ and $z\ F\ x$, but this would contradict $F \odot F^{-1} \subseteq \underline{co}_X$. The situation for ${}^\bullet x_F = \emptyset$ is similar. □

Local states or places have exactly one immediate temporal predecessor and one successor.

**Proposition P35** Let $(S, T, F) \in Nets(CS)$.
Then $\forall s \in S : |\ {}^\bullet s_X\ | = |\ s^\bullet{}_X\ | = 1$. □

**Proof** Directly from P21 and P34. □

Events or transitions are necessarily equipped with more than one predecessor and more than one successor.

**Proposition P36** Let $(S, T, F) \in Nets(CS)$.
Then $\forall t \in T : |\ {}^\bullet t_X\ | > 1 \wedge |\ t^\bullet{}_X\ | > 1$. □

**Proof** Let $t \in T$. Assume $s \in {}^\bullet t_X$ and $s' \in t^\bullet{}_X$ (this is possible by P34). Certainly we have $s\ P\ t$ and $s'\ P\ t$. P24 implies $\exists z : s\ co\ z \wedge z\ P\ t$ and $\exists z' : s'\ co\ z' \wedge z'\ P\ t$ such that D12 leads to $z \in {}^\bullet t_X$ and $z' \in t^\bullet{}_X$. □

A first hint about the cardinality of $Nets(CS)$ in given by the next proposition. If a net $N$ is contained in this set, this is also true for the inverse net of $N$. So, if $Nets(CS)$ is finite, it must be of even cardinality.

**Proposition P37** $(S, T, F) \in Nets(CS) \Leftrightarrow (S, T, F^{-1}) \in Nets(CS)$. □

**Proof** It is sufficient to show: $F$ is a consistent orientation $\Leftrightarrow F' := F^{-1}$ is a consistent orientation. And this follows directly from D12:

a) $F \cup F^{-1} = F' \cup F'^{-1} = im$,
b) $F \cap F^{-1} = F' \cap F'^{-1} = \emptyset$,
c) $F^2 \subseteq li \Leftrightarrow F'^2 \subseteq li^{-1} = li$,
d) $F \odot F^{-1} = F'^{-1} \odot F' \subseteq \underline{co}_X$,
e) $F^{-1} \odot F = F' \odot F'^{-1} \subseteq \underline{co}_X$.

□

In the following a proposition is prepared, which shows that $Nets(CS)$ must contain at least one element, if a certain condition (ASS2) is satisfied. To state this condition in a compact form and to simplify the proof two abbreviations are introduced. Given an arbitrary $im$-chain $A$ we say that (the unordered set of) two concurrent elements, which are separated by only one element in $A$, constitute a *co*-root. $CoRoots_{CS}(A)$ is simply the set of *co*-roots on $A$. If $B$ is an $im$-cycle, the idea is similar, but we have to take into account also the two neighbors of the first (which is the same as the last) element leading to the definition of $AllCoRoots_{CS}(B)$.

**Definition D15**
Let $A = (a_0, \ldots, a_n)$ be an $im$-chain and $B = (b_0, \ldots, b_n)$ be an $im$-cycle.

a) $CoRoots_{CS}(A) := \{i \in \mathbb{N}[1, n-1] : a_{i-1} \ \underline{co}_X \ a_{i+1}\}$;
b) $AllCoRoots_{CS}(B) := \{i \in \mathbb{N}[0, n-1] : b_{(i-1) \bmod n} \ \underline{co}_X \ b_{(i+1) \bmod n}\}$.

□

For an illustrating example look at Fig. 9, where we see an $im$-cycle $A = (1, 2, 3, 4, 5, 6, 7, 8, 1)$ and the following relations: $1 \ li \ 3$, $2 \ li \ 4$, $3 \ co \ 5$, $4 \ li \ 6$, $5 \ li \ 7$, $6 \ li \ 8$, $7 \ li \ 1$. Further elements and relations are not of interest. Conceiving $A$ as an $im$-chain we see that $CoRoots(A) = \{\ 3\}$. Viewing $A$ as an $im$-cycle we find the same result for $AllCoRoots(A) = \{3\}$. Choosing a different $im$-cycle $B = (4, 5, 6, 7, 8, 1, 2, 3, 4)$ (on the same set of elements) we get $CoRoots(B) = \emptyset$, but $AllCoRoots(B) = \{0\}$.

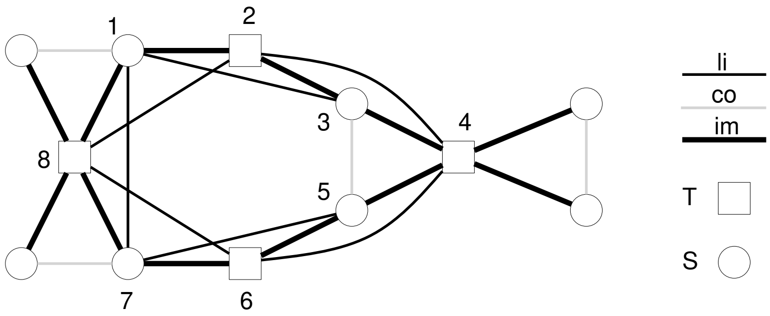


Figure 9: An $im$-cycle with odd *co*-roots

The subsequent remark should clarify the relation between these two definitions.

**Remark R14** Let $A = (a_0, \ldots, a_n)$ be an $im$-cycle.

a) $a_1 \ li \ a_{n-1} \Rightarrow AllCoRoots_{CA}(A) = CoRoots_{CS}(A)$;
b) $a_1 \ co \ a_{n-1} \Rightarrow AllCoRoots_{CA}(A) = CoRoots_{CS}(A) \cup \{0\}$;

□

Dealing with concurrency in analogy to information as a flowing quantity *co*-roots are intended to mark potential sources and sinks of concurrency. The following assumption, states that on every $im$-cycle the cardinality of *co*-roots is even. Loosely speaking, this seems to be a necessary condition for the conservation of concurrency on an $im$-cycle (whatever this may mean), if we argue that for every source there must be a sink of concurrency, and vice versa.

**Assumption ASS2** Let $A$ be an $im$-cycle.
Then $\mid AllCoRoots_{CS}(A) \mid$ is even. □

The next definition will be helpful in subsequent proofs. It simply defines the concept of consistent orientation with respect to $im$-chains similar to D12. Note that in general $im|A \neq im|Set(A)$ if $A$ is a $im$-chain. Hence the following notion is generally not equivalent to a consistent orientation of $A$.

**Definition D16** Let $F \subseteq X \times X$ and $A$ be an $im$-chain.
*$F$ is an $A$-orientation in $CS$* $:\Leftrightarrow$

a) $F \cap F^{-1} = \emptyset$;
b) $F \cup F^{-1} = im|A$;
c) $a_i\ F\ a_{i+1}\ F\ a_{i+2} \Rightarrow a_i\ li\ a_{i+2}$;
d) $a_i\ F^{-1}\ a_{i+1}\ F^{-1}\ a_{i+2} \Rightarrow a_i\ li\ a_{i+2}$;
e) $a_i\ F\ a_{i+1}\ F^{-1}\ a_{i+2} \Rightarrow a_i\ co\ a_{i+2}$;
f) $a_i\ F^{-1}\ a_{i+1}\ F\ a_{i+2} \Rightarrow a_i\ co\ a_{i+2}$.

□

Certainly a consistent orientation $F$ of $CS$ implies that $F$ is an orientation on every $im$-chain.

**Remark R15** Let $A$ be an $im$-chain.
Then $F$ is a consistent orientation on $X \Rightarrow F|A$ is an $A$-orientation. □

**Proof** Use R11. □

From the fact that $CoRoots_{CS}(A)$ is of even cardinality for a given $im$-chain $A$ we can intuitively conclude that an $A$-orientation $F$ fixed between the first two elements of the chain can be propagated along the chain and undergoes an even number of reversals (one for each *co*-root), such that at the end of the chain we again find the original orientation. If $\mid CoRoots_{CS}(A) \mid$ is odd, the last two elements of the chain are oriented in the opposite direction.

**Lemma L9** Let $A = (a_0, \ldots, a_n)$ be an $im$-chain with $n \geq 2$ and
$F$ be a $A$-orientation in $CS$.

a) $\mid CoRoots_{CS}(A) \mid$ is even $\Rightarrow a_0\ F\ a_1 \Leftrightarrow a_{n-1}\ F\ a_n$;
b) $\mid CoRoots_{CS}(A) \mid$ is odd $\Rightarrow a_0\ F\ a_1 \Leftrightarrow a_{n-1}\ F^{-1}\ a_n$.

□

**Proof** Define $r_Y := \mid CoRoots_{CS}(Y) \mid$ and proceed by induction over $r_A$. For $r_A = 0$ we have either $a_i\ F\ a_{i+1}$ for all $i, i+1 \in \mathbb{N}[0,n]$ or $a_i\ F^{-1}\ a_{i+1}$ for all $i, i+1 \in \mathbb{N}[0,n]$. So L9a is satisfied and L9b holds trivially. Now we show that our lemma is also valid for $r_A = i > 0$

if it holds for $r_A = i - 1$: For $r_A = i$ there is a pair $a_{k-1}, a_{k+1} \in A$ with $a_{k-1} \ \underline{co}_X \ a_{k+1}$. Choose the largest $k$ with this property. Now decompose $A$ into $B = (b_0, \ldots, b_n)$ and $C = (c_0, \ldots, c_m)$ such that $A = (b_0, \ldots, b_{n-1} = a_{k-1}, b_n = a_k = c_0, c_1 = a_{k+1}, \ldots, c_m)$ and notice that $r_C = 0$, $b_{n-1} \ \underline{co}_X \ c_1$ and $r_B = r_A - 1$.
If $r_A$ is odd then $r_B$ is even and we conclude $b_0 \ F \ b_1 \Leftrightarrow b_{n-1} \ F \ b_n$ (using L9a with $r_A = i - 1$) and $b_{n-1} \ F \ b_n \Leftrightarrow c_0 \ F^{-1} \ c_1$ (from R11) and $c_0 \ F \ c_1 \Leftrightarrow c_{n-1} \ F \ c_n$ (from L9a). Altogether this yields $b_0 \ F \ b_1 \Leftrightarrow c_{n-1} \ F^{-1} \ c_n$ which is actually L9b.
If $r_A > 0$ is even then $r_B$ is odd and we find $b_0 \ F \ b_1 \Leftrightarrow b_{n-1} \ F^{-1} \ b_n$ (applying L9b with $r_A = i - 1$) and $b_{n-1} \ F \ b_n \Leftrightarrow c_0 \ F^{-1} \ c_1$ (from R11) and $c_0 \ F \ c_1 \Leftrightarrow c_{n-1} \ F \ c_n$ (from L9a). Altogether this yields $b_0 \ F \ b_1 \Leftrightarrow c_{n-1} \ F \ c_n$ which is L9a. □

Combining the previous lemma with the assumption that on *im*-cycles there is always an even number of *co*-roots we can find the following lemma concerning $F$ in the neighborhood of the first and last element of an *im*-cycle.

**Lemma L10** Let $A = (a_0, \ldots, a_n)$ be an *im*-cycle with $n \geq 2$ and
$F$ be an $A$-orientation in $CS$.

a) $a_{n-1} \ li \ a_1 \Rightarrow a_{n-1} \ F \ (a_n = a_0) \ F \ a_1 \vee a_{n-1} \ F^{-1} \ (a_n = a_0) \ F^{-1} \ a_1$;
b) $a_{n-1} \ \underline{co}_X \ a_1 \Rightarrow a_{n-1} \ F \ (a_n = a_0) \ F^{-1} \ a_1 \vee a_{n-1} \ F^{-1} \ (a_n = a_0) \ F \ a_1$.

□

**Proof** First apply ASS2 to $A$ and observe that $AllCoRoots(A)$ is even. We distinguish two cases: Either $a_{n-1} \ li \ a_1$ **(1)** or $a_{n-1} \ \underline{co}_X \ a_1$ **(2)**. In case 1 we have that $CoRoots(A)$ is even and in case 2 $CoRoots(A)$ is odd (see R14). Applying L9 yields $a_0 \ F \ a_1 \Leftrightarrow a_{n-1} \ F \ a_n$ in case 1 and $a_0 \ F \ a_1 \Leftrightarrow a_{n-1} \ F^{-1} \ a_n$ in case 2. □

With these lemmas we are prepared to give a procedure to construct a consistent orientation on an arbitrary large *im*-coherent subset of $X$ (that is a subset $Y \subseteq X$ with $im^*_Y = Y$) and to prove that this procedure yields a (total) consistent orientation in the limit.

**Scope S3**

The subsequent construction is guided by an enumeration of all elements of $X$, which must have the property that it preserves the coherence of *im*, that is, the set of all elements enumerated up to a certain index should be *im*-coherent ($X$ is *im*-coherent iff $im^*_X = X \times X$). This is ensured by the additional condition that a newly enumerated element should be the *im*-neighbor of a previously enumerated one.

**Definition D17**

a) *e is an enumeration of* $CS$ $:\Leftrightarrow$
  $e : \mathbb{N} \to X$ and $e$ is surjective on $X$ and total on $\mathbb{N}$ or $\mathbb{N}[0, n]$ for some $n$;
b) *e is a coherent enumeration of* $CS$ $:\Leftrightarrow$
  $\forall i \in Dom(e) : i > 0 \Rightarrow \exists j \in Dom(e) : j < i \wedge e(j) \ im \ e(i)$.

□

The following construction inductively defines several sets, which are indexed over the domain of an arbitrary coherent and injective enumeration, and finally defines $F$ in terms of these sets (D18g). Notice that there is some further arbitrariness in this construction (as we have the possibility to choose between alternatives in particular at D18b), such that we can not exclude the possibility that the construction yields different results in terms of $F$. But this does not matter, as our first goal is only to prove the existence of at least one consistent orientation.

**Construction D18**

a) Fix a coherent injective enumeration $e$ of $CS$ and define $e_i := e(i)$;
b) Choose $D_0$, $E_0$ such that $\{D_0, E_0\} = im[e_0]/\underline{co}_X$;
c) $F_0 := DF_0 := (D_0 \times \{e_0\}) \cup (\{e_0\} \times E_0)$;
d) Choose $D_i$, $E_i$ such that $\{D_i, E_i\} = im[e_i]/\underline{co}_X$ and
$\exists j : j < i \wedge (e_j\ F_j\ e_i \wedge e_j \in D_i) \vee (e_i\ F_j\ e_j \wedge e_j \in E_i)$;
e) $DF_i := (D_i \times \{e_i\}) \cup (\{e_i\} \times E_i)$;
f) $F_i := F_{i-1} \cup DF_i$;
g) $F := \bigcup\{F_i : i \in Dom(e)\}$.

□

The proof that $F$ is well-defined by this construction will follow after some auxiliary definitions and lemmas. But first the idea behind this construction will be informally sketched.

Starting with an orientation of the neighborhood of $e_0$ we will propagate this orientation through the whole structure with the help of our enumeration $e$. In every step $i$ we orient $e_i$ and its immediate neighborhood $im[e_i]$ in some locally consistent manner by $DF_i$, and we hope that collecting all local orientations yields a (global) orientation of $X$. Let us go step by step through the construction: In D18b we simply name the two equivalence classes of $im[e_0]$ with respect to $\underline{co}_X$. By convention $D_0$ should denote the immediate temporal predecessors of $e_0$, and $E_0$ contains its immediate temporal successors. D18c establishes $F_0$ to be consistent with this choice, that is $F_0^{-1}[e_0] = D_0$ and $F_0[e_0] = E_0$. D18d similarly defines the equivalence classes of $im[e_i]$ to be $D_i$ and $E_i$, but now with the boundary condition that this choice is consistent with a previous orientation already established by $F_j$ for $j < i$: That is if $e_i$ is already chosen to be the successor of some $e_j$, then $e_j$ must be contained in the predecessors $D_i$ of $e_i$. Otherwise, if $e_i$ is already a predecessor of some $e_j$, we want $e_j$ to be contained in the successors $E_i$ of $e_i$. D18e again expresses the convention that $D_i$ and $E_i$ are predecessors and successors, respectively. D18f collects all local orientations $DF_j$ in $F_i$, which have be found up to and including step $i$. Finally D18g yields the smallest set $F$ containing all $F_j$ constructed during the procedure, which may be (countably) infinitely many.

The subsequent list of properties concerning the previous construction can be easily verified.

**Remark R16**

a) $i \leq j \Rightarrow DF_i \subseteq F_j$;
b) $i \leq j \Rightarrow F_i \subseteq F_j$;
c) $F_i \subseteq F$;
d) $DF_i \cap DF_i^{-1} = \emptyset$;
e) $DF_i^2 \subseteq li$;
f) $DF_i \odot DF_i^{-1} \subseteq \underline{co}_X$;
g) $DF_i^{-1} \odot DF_i \subseteq \underline{co}_X$;
h) $DF_i \cup DF_i^{-1} =\updownarrow (im[e_i] \times \{e_i\})$;
i) $DF_i \cup DF_i^{-1} = im|(\underline{im}_X[e_i])$;

j) $DF_i \subseteq im$;
k) $F \cup F^{-1} \subseteq im$;
l) $F_i = \bigcup\{DF_j : j \in \mathbb{N}[0,i]\}$;
m) $e_i\ DF_j\ e_k \Rightarrow i = j \vee j = k$.

□

In the next definition $X_i$ is the set of all elements and their $im$-neighbors reached by the enumeration up to and including step $i$. Notice that the intention of a coherent enumeration was just to guarantee that $X_i$ is $im$-coherent.

**Definition D19** $X_i := \bigcup\{\underline{im}_X[e_k] : k \in \mathbb{N}[0,i]\}$. □

The infinite sequence $X_0, X_1, \ldots$ is a subset chain and $X$ is the smallest set containing all $X_i$.

**Remark R17**

a) $e_0, \ldots, e_i \in X_i$;
b) $e_i\ im\ e_j \Rightarrow e_i \in X_j \wedge e_j \in X_i$;
c) $X = \bigcup\{X_i : i \in N\}$;
d) $i \leq j \Rightarrow X_i \subseteq X_j$.

□

By the help of $X_i$ we recognize a further important property of $DF_i$ and $F_i$: $X_i$ is exactly that subset of $X$ up to which the orientation has been propagated though the structure up to step $i$.

**Lemma L11**

a) $(x\ DF_i\ y \vee x\ DF_i^{-1}\ y) \Rightarrow x, y \in X_i$;
b) $F_i \cup F_i^{-1} = im|X_i$;
c) $F_i \cup F_i^{-1} = (F_i \cup F_i^{-1})|X_i$.

□

**Proof**

a) Assume $e_k\ DF_i\ e_j$ or $e_k\ DF_i^{-1}\ e_j$. It follows that $e_k\ im\ e_j$ which implies $e_k \in X_j$ and $e_j \in X_k$ because $e_j \in X_j$ and $e_k \in X_k$. By D18e we have $DF_i = (D_i \times \{e_i\}) \cup (\{e_i\} \times E_i)$ which means that either $k = i$ or $j = i$. So we conclude that $e_k \in X_i$ and $e_j \in X_i$ in both cases.
b) By R16l $F_i \cup F_i^{-1} = \bigcup\{DF_j \cup DF_j^{-1} : j \in \mathbb{N}[0,i]\}$ and by R16i $DF_i \cup DF_i^{-1} = im|(\underline{im}_X[e_i])$. It follows that $F_i \cup F_i^{-1} = \bigcup\{im|(\underline{im}_X[e_j]) : j \in \mathbb{N}[0,i]\} = im|X_i$.
c) From L11b by $(F_i \cup F_i^{-1})|X_i = (im|X_i)|X_i = im|X_i = F_i \cup F_i^{-1}$.

□

As promised, it follows the proof that $F$ is well-defined in our construction, which means in this case that the construction always succeeds to yield some $F$. This is not immediately clear, as, in particular, it is not evident that the condition of D18d can be always satisfied.

**Lemma L12**

a) $D_i$,$E_i$ are well-defined;

b) $DF_i$,$F_i$,$F$ are well-defined.

□

**Proof**

a) Remember P20. It guarantees that we can always choose $D_i$,$E_i$ such that $\{D_i, E_i\} = im[e_i]/\underline{co}_X$. But what about the boundary condition in D18d: $\exists j : j < i \wedge (e_j\ F_j\ e_i \wedge e_j \in D_i) \vee (e_i\ F_j\ e_j \wedge e_j \in E_i)$. First in D17b we required that there is some $j$ such that $j < i \wedge e_j\ im\ e_i$. This means that $e_i \in X_j$ and of course $e_j \in X_j$. By L11b ($F_j \cup F_j^{-1} = im|X_j$) we have $e_i\ F_j\ e_j$ or $e_j\ F_j\ e_i$. Furthermore $e_j \in im[e_i]$ such that in case $e_j\ F_j\ e_i$ we can choose $D_i$ such that $e_j \in D_i$ and otherwise we must have $e_i\ F_j\ e_j$ and we choose $E_i$ such that $e_j \in E_i$. Hence the condition in D18d can always be satisfied.

b) Follows immediately from L11c as $F$,$F_i$,$DF_i$ are defined in terms of $D_i$,$E_i$.

□

Unfortunately, the straightforward proof that the result of construction D18 is indeed a consistent orientation, which follows now, is technical and might obscure the actual idea, which is the following: If every $im$-cycle has an even number of $co$-roots, it can be oriented consistently. If every $im$-cycle can be oriented consistently, this is also true for the whole set $X$. That $im$-cycles have to be considered comes from the observation that, if a concurrency structure exists, which contains the structure of Fig. 9 as a subset (remember that this was assumed not to be the case by ASS2, as we have exactly one $co$-root on the $im$-cycle shown), it is impossible to find a consistent orientation.

The following two lemmas prove the first two necessary conditions D12b and D12a for $F_i$ to be a consistent orientation on $X_i$.

**Lemma L13** $\forall x, y \in X_i : (x\ im\ y \Leftrightarrow x\ F_i\ y \vee x\ F_i^{-1} y)$. □

**Proof** This is equivalent to $im|X_i = (F_i \cup F_i^{-1})|X_i$. With $(F_i \cup F_i^{-1})|X_i = F_i \cup F_i^{-1}$ (L11c) we find $F_i \cup F_i^{-1} = im|X_i$ which is L11b. □

The proof of the next lemma is simple but cumbersome, as there are several cases, that have to be distinguished. The proof is an indirect one and has roughly the following structure: First we identify the first step in our construction that violates our lemma. We show that there must exist at least two $im$-paths from $e_0$ to that location, where the violation takes place. Intuitively these to paths represent contradicting orientation constraints carried to an element from different sides. Combining these two $im$-paths to a cycle we apply ASS2 (or L10 to be more precise) to derive a contradiction.

**Lemma L14** $\forall x, y \in X_i : \neg(x\ F_i\ y \wedge x\ F_i^{-1}\ y)$. □

**Proof** Indirect: Assume $F_i \cap F_i^{-1} \neq \emptyset$ for some $i$. We will choose the smallest such $i$ in order to ensure that $F_{i-1} \cap F_{i-1}^{-1} = \emptyset$. This means there are $e_j, e_k \in X_i$ such that $e_j\ F_i\ e_k$ and $e_k\ F_i\ e_j$. As $DF_i \cap DF_i^{-1} = \emptyset$ we can distinguish two cases: Either $e_j\ F_{i-1}\ e_k$ **(1)** or $e_k\ F_{i-1}\ e_j$ **(2)**. Observe that in both cases we have $e_j, e_k \in X_{i-1}$ **(3)** by L11b.
First we deal with case 1: If $e_j\ F_{i-1}\ e_k$ then there is some $l$ with $l < i$ **(4)** and $e_j\ DF_l\ e_k$, which implies either $l = j$ **(5)** or $l = k$ **(6)**. Now consider $e_k\ F_i\ e_j$. From $e_j\ F_{i-1}\ e_k$ and $F_{i-1} \cap F_{i-1}^{-1} = \emptyset$ we infer $e_k\ DF_i\ e_j$, which implies either $i = j$ **(7)** (if $e_k \in D_i$) or $i = k$ **(8)** (if $e_j \in E_i$).

Now we come to the case 2, which is completely symmetric to 1: $e_k\ F_{i-1}\ e_j$ implies that there is some $l$ with $l < i$ **(9)** and $e_k\ DF_l\ e_j$. Again this implies either $l = j$ **(10)** or $l = k$ **(11)**. From $e_j\ F_i\ e_k$ together with $e_k\ F_{i-1}\ e_j$ and $F_{i-1} \cap F_{i-1}^{-1} = \emptyset$ we derive $e_j\ DF_i\ e_k$. This means either $i = j$ **(12)** (if $e_k \in E_i$) or $i = k$ **(13)** (if $e_j \in D_i$) holds.
For the case 1 as well as for the case 2 we found that $(e_k, e_j) \in\updownarrow DF_i$ such that the condition of D18d must have been satisfied: There is some $m$ with $m < i$ and $e_m\ F_m\ e_i \wedge e_m \in D_i$ **(14)** or $e_i\ F_m\ e_m \wedge e_m \in E_i$ **(15)**.
Our aim is to derive a contradiction for every conceivable combination of cases introduced above:

a) For $(5 \wedge 7)$, $(6 \wedge 8)$, $(10 \wedge 12)$ and $(11 \wedge 13)$ this is easily done: These combinations are impossible because they imply $i = l$ which contradicts either 4 or 9.
b) $(1 \wedge 5 \wedge 8 \wedge 14)$. $e_j\ F_{i-1}\ e_k$, $l = j$, $i = k$, $e_m\ F_m\ e_i$.
$e_m \in D_i$ and $(e_j = e_l) \in E_i$ implies $e_m\ li\ (e_j = e_l)$.
c) $(1 \wedge 6 \wedge 7 \wedge 14)$. $e_j\ F_{i-1}\ e_k$, $l = k$, $i = j$, $e_m\ F_m\ e_i$.
$e_m \in D_i$ and $(e_k = e_l) \in D_i$ implies $e_m\ co\ (e_k = e_l)$.
d) $(1 \wedge 5 \wedge 8 \wedge 15)$. $e_j\ F_{i-1}\ e_k$, $l = j$, $i = k$, $e_i\ F_m\ e_m$.
$e_m \in E_i$ and $(e_l = e_j) \in E_i$ implies $(e_l = e_j)\ co\ e_m$.
e) $(1 \wedge 6 \wedge 7 \wedge 15)$. $e_j\ F_{i-1}\ e_k$, $l = k$, $i = j$, $e_i\ F_m\ e_m$.
$e_m \in E_i$ and $(e_k = e_l) \in D_i$ implies $e_m\ li\ (e_k = e_l)$.
f) $(2 \wedge 10 \wedge 13 \wedge 14)$. $e_k\ F_{i-1}\ e_j$, $l = j$, $i = k$, $e_m\ F_m\ e_i$.
$e_m \in D_i$ and $(e_j = e_l) \in D_i$ implies $e_m\ co\ (e_j = e_l)$.
g) $(2 \wedge 11 \wedge 12 \wedge 14)$. $e_k\ F_{i-1}\ e_j$, $l = k$, $i = j$, $e_m\ F_m\ e_i$.
$e_m \in D_i$ and $(e_k = e_l) \in E_i$ implies $e_m\ li\ (e_k = e_l)$.
h) $(2 \wedge 10 \wedge 13 \wedge 15)$. $e_j\ F_{i-1}\ e_k$, $l = k$, $i = j$, $e_m\ F_m\ e_i$.
$e_m \in E_i$ and $(e_j = e_l) \in D_i$ implies $e_m\ li\ (e_j = e_l)$.
i) $(2 \wedge 11 \wedge 12 \wedge 15)$. $e_k\ F_{i-1}\ e_j$, $l = k$, $i = j$, $e_i\ F_m\ e_m$.
$e_m \in E_i$ and $(e_k = e_l) \in E_i$ implies $e_m\ co\ (e_k = e_l)$.

Except for the first combination (which was already found to be impossible) we observe (using the fact that $F_m \subseteq F_{i-1}$ from $m < i$):

a) $e_m\ co\ e_l \Leftrightarrow e_m\ F_{i-1}\ e_i\ F_{i-1}\ e_l \vee e_m\ F_{i-1}^{-1}\ e_i\ F_{i-1}^{-1}\ e_l$ **(16)**;
b) $e_m\ li\ e_l \Leftrightarrow e_m\ F_{i-1}\ e_i F_{i-1}^{-1}\ e_l \vee e_m\ F_{i-1}^{-1}\ e_i F_{i-1}\ e_l$ **(17)**.

As $e_l, e_m, e_i \in X_{i-1}$ (remember 3 and $m < i$) there are two $(im|X_{i-1})$-chains $A$,$B$ with $A = (e_i, e_l, \ldots, e_0)$ and $B = (e_i, e_m, \ldots, e_0)$. Exploiting the condition of D17b we can even choose $A$,$B$ such that $p < q \wedge A_p = e_r \wedge A_q = e_s \Rightarrow r > s$ and $p < q \wedge B_p = e_r \wedge B_q = e_s \Rightarrow r > s$. Combining $A$ and $B$ we construct a $(im|X_{i-1})$-cycle $C = (c_n)_{n \in I} = (e_i, e_l, \ldots, e_0, \ldots, e_m, e_i)$. Note that $i$ is the maximal index necessary to construct $C$, that is, $\neg\exists p > i : e_p \in C$ **(18)**.

Now define $F' := F_{i-1}|C$ and notice that it has the following properties:

a) $F' \cap F'^{-1} = \emptyset$. By our initial assumption $F_{i-1} \cap F_{i-1}^{-1} = \emptyset$.
b) $F' \cup F'^{-1} = im|C$. Use L11b.
c) $c_n\ F'\ c_{n+1}\ F'\ c_{n+2} \Rightarrow c_n\ li\ c_{n+2}$. To see this assume there are $e_p, e_q, e_r \in C$ with $e_p\ F'\ e_q\ F'\ e_r$ and $e_q \neq e_i$ but $e_p\ co\ e_r$. It is immediately clear that $q \neq i$. By our construction we have either $e_p\ DF_q\ e_q\ DF_q^{-1}\ e_r$ or $e_p\ DF_q^{-1}\ e_q\ DF_q\ e_r$. If $q < i$ we

get a contradiction with $F_{i-1} \cap F_{i-1}^{-1} = \emptyset$, because $DF_q \subseteq F_{i-1}$. So we are left with $q > i$, but this is impossible, because $i$ is the maximal index appearing in $C$ (see 18).

d) $c_n \; F'^{-1} \; c_{n+1} \; F'^{-1} \; c_{n+2} \Rightarrow c_n \; li \; c_{n+2}$. The proof is similar to the previous one.

e) $c_n \; F' \; c_{n+1} \; F'^{-1} \; c_{n+2} \Rightarrow c_n \; co \; c_{n+2}$. Assume $e_p, e_q, e_r \in C$ with $e_p \; F' \; e_q \; F'^{-1} \; e_r$, $e_p \neq e_r$ and $e_q \neq e_i$ but $e_p \; li \; e_r$. It is clear that $q \neq i$. According to our construction we have either $e_p \; DF_q \; e_q \; DF_q \; e_r$ or $e_r \; DF_q \; e_q \; DF_q \; e_p$. As above $q < i$ and $q > i$ lead to contradiction.

f) $c_n \; F'^{-1} \; c_{n+1} \; F' \; c_{n+2} \Rightarrow c_n \; co \; c_{n+2}$. The proof is similar to the previous one.

With these facts we immediately conclude that $F'$ is a $C$-orientation, and, as $C$ is an $im$-cycle, we can apply L10, which states that in case of $c_{n-1} \; li \; c_1$ we have $c_{n-1} \; F' \; (c_n = c_0) \; F' \; c_1 \vee c_{n-1} \; F'^{-1} \; (c_n = c_0) \; F'^{-1} \; c_1$ contradicting 17 and in case of $c_{n-1} \; co \; c_1$ we have $c_{n-1} \; F' \; (c_n = c_0) \; F'^{-1} \; c_1 \vee c_{n-1} \; F'^{-1} \; (c_n = c_0) \; F' \; c_1$ contradicting 16. □

Now it is left to prove D12c and D12d, which can be done directly for $F$ with respect to $X$ by the next two lemmas.

**Lemma L15** $x \; im \; y \; im \; z \wedge x \; li \; z \Rightarrow x \; F \; y \; F \; z \vee x \; F^{-1} \; y \; F^{-1} \; z$. □

**Proof** For $e_l, e_j, e_k \in X_i$ assume $e_l \; im \; e_j \; im \; e_k$ and $e_l \; li \; e_k$. First it is clear that $l \neq j$,$j \neq k$ and $l \neq k$. By D18e we have $DF_j = (D_j \times \{e_j\}) \cup (\{e_j\} \times E_j)$ with $\{D_j, E_j\} = im[e_j]/\underline{co}_X$. $e_l \; li \; e_k$ requires either $e_l \in D_j \wedge e_k \in E_j$ or $e_k \in D_j \wedge e_l \in E_j$. Hence either $e_l \; DF_j \; e_j \; DF_j \; e_k$ or $e_k \; DF_j \; e_i \; DF_j \; e_l$ holds and $DF_j \subseteq F$ proves our lemma. □

**Lemma L16** $x \; im \; y \; im \; z \wedge x \; co \; z \Rightarrow x \; F \; y \; F^{-1} \; z \vee x \; F^{-1} \; y \; F \; z$. □

**Proof** For $e_l, e_j, e_k \in X_i$ assume $e_l \; im \; e_j \; im \; e_k$ and $e_l \; co \; e_k$. As above $l \neq j$,$j \neq k$ and $l \neq k$. Now D18e yields $DF_j = (D_j \times \{e_j\}) \cup (\{e_j\} \times E_j)$ with $\{D_j, E_j\} = im[e_j]/\underline{co}_X$. $e_l \; co \; e_k$ implies either $e_l, e_k \in D_j$ or $e_l, e_k \in E_j$. This means either $e_l \; DF_j \; e_j \; DF_j^{-1} \; e_k$ or $e_l \; DF_j^{-1} \; e_j \; DF_j \; e_k$ holds and with $DF_j \subseteq F$ this completes the proof. □

Finally we can state our first important conclusion that $F$ is indeed a consistent orientation.

**Lemma L17** $F$ is a consistent orientation on $CS$. □

**Proof** Finite or transfinite induction over $i \in Dom(e)$ with $F = \bigcup\{F_i : i \in Dom(e)\}$ applied to L13 and L14 yields $F \cup F^{-1} = im$ and $F \cap F^{-1} = \emptyset$. In combination with L15 and L16 this shows that $F$ is a consistent orientation on $X$ as D12b is satisfied. □

□ S3

This directly implies that we can find at least one net associated with an arbitrary concurrency structure.

**Corollary C8** $\mid Nets(CS) \mid \geq 1$. □

**Proof** We have $(S, T, F) \in Nets(CS)$ with $F$ constructed in D18 as consistent orientation (L17). □

As with every consistent orientation $F$ we have immediately a further consistent orientation $F^{-1}$ there must be at least two nets in $Nets(CS)$, and we can even prove that there cannot be more: This can be derived from the fact that every concurrency structure is coherent, such that local choice of the orientation between two neighbors propagates through the whole structure and determines the (total) orientation uniquely.

**Lemma L18** $\mid Nets(CS) \mid \leq 2$. □

**Proof** Assume $\mid Nets(CS) \mid > 2$. Then there are $F, F^{-1}, F' \in Nets(CS)$ with $F' \neq F \wedge F' \neq F^{-1}$. Hence there are $a, b, c, d \in X$ with $a\ F\ b \wedge c\ F\ d$ and $b\ F'\ a \wedge c\ F'\ d$. By P7 it is clear that there is an $im$-chain $A = (a_0, \ldots, a_n)$ with $a_0 = a \wedge a_1 = b \wedge a_{n-1} = c \wedge a_n = d$. Furthermore $CoRoots(A)$ is either even or odd. So applying L9 to $F$ and $F'$ (notice that this is possible due to R15) yields either $a\ F\ b \Leftrightarrow c\ F\ d$ and $a\ F'\ b \Leftrightarrow c\ F'\ d$ or $a\ F\ b \Leftrightarrow c\ F^{-1}\ d$ and $a\ F'\ b \Leftrightarrow c\ F'^{-1}\ d$. In both cases we have a contradiction with our initial choice of $a, b, c, d$. □

**Proposition P38** $\mid Nets(CS) \mid = 2$. □

**Proof** By C8, L18 and P37. □

Altogether, the formalism of nets has several advantages compared to concurrency structures which include the explicit differentiation between active and passive elements, the explicit choice of an arrow of time, which simplifies reasoning and the reduction of the descriptional complexity (generally the flow relation is much smaller than causality or concurrency relations). Later we will recognize a further convenience of the net formalism concerned with the meaning of the token game in elementary net systems.

One might wonder why concurrency theory does not choose the level of nets as a basis to formulate the axioms. One reason is that from the viewpoint of constructing an axiomatic system it is natural to identify atomic notions (that are notions of which is is believed that a further regress to simpler concepts is impossible or not appropriate). An additional advantage emerges, if these notions are easy to understand and can be found on very different levels of abstraction such that there is no dissent about they properties. In the case of concurrency theory these atomic notions are concurrency and causality defined on some uniform set. To assume as less as possible concerning the structure carrying the axioms helps to make every postulate explicit by an additional axiom (as simple as it might be). It should be the set of axioms, which then leads to the actual complexity of the matter. That the field reveals a certain simplicity by the use of a convenient notation (as it seems to be for concurrency theory on the level of nets and elementary net systems) might be a hint to practical applicability of the theory.

## 5.9 The Flow Relation in Cyclic and Acyclic Structures

The flow relation with its property to be a consistent orientation is a useful vehicle to reason about concurrency structures, as it provides a lingual means to refer to the past or future with respect to a given element.

The first important property of the flow relation arises from the fact that lines are $im$-coherent. As one might already expect every line is oriented by the flow relation in exactly one direction. This is implied by the following proposition, which states that given two elements $x, y$ on a line they are connected by a chain of the flow relation.

**Proposition P39** Let $l \in Lines(CS)$ and $A$ be an acyclic $im|l$-chain.
Then $A$ is either an $F|l$-chain or an $F^{-1}|l$-chain. □

**Proof** Choose $x, y \in l$ arbitrarily. Define $AC(x, y) := \{C : \mid C \mid > 1 \wedge C = (x, \ldots, y)$ is an acyclic $im|l$-chain $\}$. Now we proceed by induction over the size of some $C \in AC(x, y)$

to prove $H(C) :\Leftrightarrow$ ($C$ is either an $F$-chain or an $F^{-1}$-chain).
For $\mid C \mid = 2$ we have $C = (x, y)$ and either $x\ F\ y$ of $y\ F\ x$ (remember D12a and D12b).
For $\mid C \mid = n$ we assume that $H(C')$ is already proved for all $C \in AC(x, y)$ with $\mid C' \mid < n$.
So we can write $C = (x = x_0, \ldots, x_{n-2}, x_{n-1})$ where $C' = (x = x_0, \ldots, x_{n-2})$ is either an $F$-chain or an $F^{-1}$-chain. In the former case it is impossible that $x_{n-1}\ F\ x_{n-2}$ as this together with $x_{n-3}\ F\ x_{n-2}$ violates D12d. So we have to conclude $x_{n-2}\ F\ x_{n-1}$ such that $C$ is actually an $F$-chain. Otherwise, if $C'$ is an $F^{-1}$-chain, a similar argument yields $x_{n-2}\ F^{-1}\ x_{n-1}$ such that $C$ is an $F^{-1}$-chain. This proves $H(C)$. □

**Corollary C9** Let $l \in Lines(CS) \wedge (S, T, F) \in Nets(CS)$.
Then $(F|l)_l^* \cup (F^{-1}|l)_l^* = l \times l$. □

**Proof** It is sufficient to prove: For every pair $x, y \in l$ we have $x\ (F|l)_l^*\ y$ or $y\ (F|l)_l^*\ x$. This is trivial for $x = y$ and follows from D6j and P39 if $x \neq y$. □

Given two elements $x$ and $y$ with $x\ li\ y$ there is certainly a line $l \in Lines(CS)$ with $x, y \in l$. Then by the above proposition there is an $F$-chain $(x, \ldots, y)$ or $(y, \ldots, x)$. In the former case we say that $x$ occurs before $y$ ($x$ is located in the past of $y$ and in the later case $x$ occurs after $y$ ($x$ is located in the future of $x$). Notice that in general these two possibilities are not mutually exclusive, such that these concepts are not always helpful as past and future may have a nonempty intersection (which will be the case for cyclic structures to be introduced below) and may even cover the whole set $X$.

Given a particular line the immediate temporal successor and predecessor exists and is unique on that line. This is what the following lemma states.

**Lemma L19** Let $l \in Lines(CS) \wedge (S, T, F) \in Nets(CS) \wedge y \in l$.
Then $\exists_1 x, z \in l : x\ F\ y\ F\ z$. □

**Proof** By P23 we can define $x, z$ by $\{x, z\} = (im|l)[y]$. Of course, we have $x, y, z \in l$. By R11 there are two possibilities: Either $x\ F\ y\ F\ z$ or $z\ F\ y\ F\ x$. So we have (exchanging $x$ and $z$ in the latter case) $\exists x, z \in l : x\ F\ y\ F\ z$. It is left to show that $x$ and $z$ are unique. Assume there is a further $x' \in l$ with $x'\ F\ y$. With D12 this implies $x'\ co\ x$ which is a contradiction with $x, x' \in l$. Similarly, the existence of a further $z' \in l$ implies $z'\ co\ z$, again a contradiction. □

For every infinite line there is an infinite, acyclic $F$-chain (more precisely, it is $\omega\omega$-infinite) that is exactly covered by that line. For a finite line, on the other hand, we can always find an $F$-cycle, such that every element of that line is contained in it.

**Proposition P40** Let $l \in Lines(CS) \wedge (S, T, F) \in Nets(CS)$.

a) $l$ is infinite $\Rightarrow \exists A : A$ is an $\omega\omega$-infinite acyclic $F$-chain $\wedge\ Set(A) = l$;
b) $l$ is finite $\Rightarrow \exists A : A$ is an $F$-cycle $\wedge\ Set(A) = l$.

□

**Proof** We will construct an $\omega\omega$-infinite $F$-chain $A = (\ldots, a_{-1}, a_0, a_1, \ldots)$. We start with $i = j = 0$ choosing some arbitrary $a_i = a_j \in l$. L19 gives us a unique successor $a_{i+1} \in l$ with $a_i\ F\ a_{i+1}$ and a unique predecessor $a_{j-1} \in l$ with $a_{j-1}\ F\ a_j$. Carrying out this step for all $i, j \in \mathbb{N}$ yields an $A$, which satisfies $Set(A) = l$, as due to C9 for every element $x \in l$ there is a finite $F$- or $F^{-1}$-chain from $a_0$ to $x$. As predecessors and successors are unique (according to L19), $A$ is either an infinite acyclic $F$-chain, or $A$ is not acyclic, which means it contains a (finite) cycle $B$ with $Set(B) = l$. If $l$ is infinite, the later case is excluded, so $A$ must be an acyclic $F$-chain. If $l$ is finite the former case is impossible, so $B$ is the $F$-cycle required by the second part of our proposition. $\square$

The preceding observation that finite lines correspond to $F$-cycles and infinite lines can be conceived as infinite, acyclic $F$-chains leads to the following natural definition: A concurrency structure is cyclic, if and only if all lines are finite, and acyclic, if and only if all lines are infinite.

**Definition D20**

a) $CS$ *is cyclic* $:\Leftrightarrow \forall l \in Lines(CS) : l$ is finite;
b) $CS$ *is acyclic* $:\Leftrightarrow \forall l \in Lines(CS) : l$ is infinite.

$\square$

An immediate question is: Are there partially cyclic concurrency structures, that is, structures, where some but not all lines are finite? It will be immediately shown that they do not exist. This property is deeply connected with the safety of the dynamics, and it will be necessary to derive the reachability results in the subsequent section.

First a lemma is proposed which shows that it is sufficient to have one finite line to ensure that the whole structure is finite. The proof is not difficult as the strong axiom D6g, which states every element has a finite concurrency neighborhood, can be exploited. An interesting question is, if it is possible to derive the same result from weaker assumptions, e.g. the finiteness of all cuts, as this might lead to an even more general formulation of concurrency theory.

**Lemma L20** $CS$ is not acyclic $\Rightarrow$ $CS$ is finite. $\square$

**Proof** Assume $CS$ is not acyclic. Then there is some finite line $l \in Lines(CS)$. Furthermore, we have $Cuts(CS) = \{c : c \cap l = \{x\} \wedge x \in l\}$ by K-density P4. Obviously $c \cap l = \{x\}$ implies $c \subseteq \underline{co}_X[x]$ which is finite by D6g. So $Cuts(CS)$ must be also finite, as $l$ is finite. Using $X = \bigcup Cuts(CS)$ we find that even $X$ is finite, as every $c \in Cuts(CS)$ is finite by R3. $\square$

An essential result that the notions of cyclic and finite structures are equivalent follows directly from this lemma.

**Remark R18** $CS$ is cyclic $\Leftrightarrow$ $CS$ is finite. $\square$

**Proof** $CS$ is finite $\Rightarrow$ $CS$ is cyclic: If $CS$ is finite, all lines are finite.
$CS$ is cyclic $\Rightarrow$ $CS$ is finite: If $CS$ is cyclic, it is not acyclic and L20 can be applied. $\square$

The anticipated fact that partially cyclic structures cannot exist is equivalent to the observation that the notions of acyclic and cyclic structures are complementary.

**Proposition P41** $CS$ is acyclic $\Leftrightarrow$ $CS$ is not cyclic. $\square$

**Proof** $CS$ is cyclic $\Rightarrow$ $CS$ is not acyclic: If $CS$ is cyclic all lines $l \in Lines(CS)$ are finite and there is no infinite line. Hence $CS$ is not acyclic.
$CS$ is not acyclic $\Rightarrow$ $CS$ is cyclic: With L20 we derive that $X$ is finite, such that of course all line $l \in Lines(CS)$ are finite, too. $\square$

Given an $F$-cycle in a cyclic concurrency structure we assume that this $F$-cycle meets every cut at least once. This property does not follow directly from K-density, as not every $F$-cycle must correspond to a line. Actually, an $F$-cycle might intersect a particular cut several times, which means, of course, that it cannot be a line in this case.

**Assumption ASS3** Let $A$ be an $F$-cycle and $c \in Cuts(CS)$.
Then $Set(A) \cap c \neq \emptyset$. □

The idea of the following lemma is similar to L20 exploiting ASS3 instead of K-density.

**Lemma L21** $\exists A : A$ is an $F$-cycle $\Rightarrow CS$ is finite. □

**Proof** Let $A$ be an arbitrary $F$-cycle. By ASS3 for every cut $c \in Cuts(CS)$ there is some $x \in c \cap Set(A)$. As $c \subseteq \underline{co}_X[x]$ is finite by D6g, $X = \bigcup Cuts(CS)$ and $Cuts(CS) = \{c : c \cap Set(A) \neq \emptyset\}$ by ASS3, we conclude that $X$ is finite. □

With this lemma it is possible to derive the next proposition, which gives a further justification for the definition D20 providing a direct link to corresponding notions in the formalism of nets.

**Proposition P42** Let $N \in Nets(CS)$.

a) $N$ is acyclic $\Leftrightarrow CS$ is acyclic;
b) $N$ is cyclic $\Rightarrow CS$ is cyclic.

□

**Proof** $N$ is acyclic $\Rightarrow CS$ is acyclic: Let $N$ be acyclic and assume $CS$ is not acyclic. Then P41 indicates that $CS$ is cyclic. Hence it must contain some $F$-cycle $A$ (by P40). But this contradicts the assumption that $N$ is acyclic.

$CS$ is acyclic $\Rightarrow N$ is acyclic: Due to P41 we can equivalently prove: $N$ is not acyclic $\Rightarrow$ $CS$ is cyclic. If $N$ is not acyclic there is some $F$-cycle and L21 suggests that $CS$ is finite, which implies that $CS$ is cyclic by R18.

$N$ is cyclic $\Rightarrow CS$ is cyclic: If $N$ is cyclic, it is not acyclic and we can use the same argument as for the previous implication. □

We conjecture that it is even possible to prove the converse of the second part of the proposition, but this will be not necessary for the subsequent line of reasoning.

With the following definition we prepare a useful notation that will be employed, when the subject of reachability of cuts by dynamical evolution is addressed. For a clique $c$ we want to denote the set of those elements, which are concurrent to all elements in $c$, by $CO(c)$.

**Definition D21** Let $c$ be a clique of $\underline{co}_X$.
$CO(c) := \bigcap\{co[x] : x \in c\}$. □

**Remark R19** Let $c$ be a clique of $\underline{co}_X$.

a) $CO(c) \cap c = \emptyset$;
b) $CO(c)$ is finite;
c) $c \in Cuts(CS) \Rightarrow CO(c) = \emptyset$.

□

**Proof** That $CO(c)$ is finite follows from D6g. □

Intuitively, if we partially fix a cut in $c$ then $CO(c)$ describes the degree of freedom of that part of the cut, which might be evolved using the concurrency propagation rules. Expressing this restriction in other words, every cut containing $c$ must be covered by $c \cup CO(c)$. Interestingly, every cut $c$ has a finite degree of freedom $CO(c)$.

The following assumption is essential, when we want to propagate a cut containing a subset $c$, which remains fixed. We have seen in C9 that every causality relation between two elements is accompanied by an $F$- or $F^{-1}$-chain between these two elements. The first part of this assumption states an even stronger condition, which is that for any two elements in $CO(c)$, that are is causality relation, there is an $F$- or $F^{-1}$ chain between them, which is completely covered by $CO(c)$. This means the causality is realized by an $F$-chain within the degree of freedom. On the other hand and this is the second part of the assumption, if $x$ and $y$ are identical or concurrent, such an $F$-chain within $CO(c)$ has to be excluded: This corresponds to the situation that for a cut that is partially frozen in $c$ two concurrent or identical elements should not be reachable by an $F$-chain within the degree of freedom $CO(c)$.

**Assumption ASS4** Let $c$ be a clique of $\underline{co}_X \wedge x, y \in CO(c)$.

a) $x \; li \; y \Rightarrow$
   $\exists A : A = (x, \ldots, y)$ or $A = (y, \ldots, x)$ is an $F|CO(c)$-chain;
b) $x \; \underline{co}_X \; y \Rightarrow$
   $\neg\exists A : A = (x, \ldots, y)$ or $A = (y, \ldots, x)$ is an $F|CO(c)$-chain $\wedge \; size(A) > 1$.

□

It immediately follows that every $F$-chain contained in $CO(c)$ must be a clique of $\underline{li}_X$.

**Remark R20** Let $c$ be a clique of $\underline{co}_X$ and $A$ be an $F|CO(c)$-chain.
Then $A$ is a clique of $\underline{li}_X$. □

**Proof** If there are $x, y \in A$ with $x \; co \; y$ we would have a contradiction with ASS4b. □

With the last assumption it is possible to prove that all nets associated with a concurrency structure are simple.

**Proposition P43** Let $N \in Nets(CS)$. Then $N$ is simple. □

**Proof** Imagine $N$ is not simple. Then there are two elements $x, x' \in X$ with $x' \neq x$ and ${}^\bullet x_F = {}^\bullet x'_F$ and $x^\bullet{}_F = x'^\bullet{}_F$. First note that $x, x' \in T$ is impossible by P35 which excludes branched $S$-elements. So we conclude that $x, x' \in S$ which means that we can find $t, t' \in T$ with $x, x' \in t^\bullet{}_F$ and $x, x' \in {}^\bullet t'_F$. Certainly $s \; co \; s'$ holds (remember D12d and D12e). By D6e there must be a $z$ with $s \; co \; z$ and $s' \; li \; z$ (without loss of generality). As $s \; li \; z$ and $s, z \in co[s']$ we can apply ASS4a which requires the existence of an $F|co[s']$-chain $A = (s, \ldots, z)$ or $A = (z, \ldots, s)$. But in any case we must have $im[s] \cap Set(A) \neq \emptyset$ such that either $t \in A$ or $t' \in A$. But this is not reconcilable with an $F|co[s']$ because $t, t' \in li[s]$.
□

### 5.10 Acyclic Concurrency Structures

It was mentioned that the conventional way to deal with concurrency and causality is based on partially ordered sets $(X, <)$ where the causality relation is derived from $\leq$ by

$li = < \cup <^{-1}$ and concurrency is the irreflexive complement. That every acyclic concurrency structure can be represented in the formalism of posets is proved in the following proposition. Given a consistent orientation $F$ the corresponding poset is simply $(X, F^*_X)$. The fact that the converse poset also represents a concurrency structure coincides with our result that there are two consistent orientations $F$ and $F^{-1}$.

**Proposition P44** Let $CS$ be acyclic and $(S,T,F) \in Nets(CS)$.
Then $li = F^+ \cup (F^{-1})^+$. $\square$

**Proof** Assume the proposition does not hold. Then there is an acyclic $F$-chain $A = (x = a_0, \ldots, a_n = y)$ with $x\ co\ y$. Let us take the shortest one. It follows that $Set(A) - \{x\}$ and $Set(A) - \{y\}$ are cliques of $\underline{li}_X$. This implies $x, y \in S$, otherwise we could apply P25 to $x$ or $y$ such that we find a shorter $A$. Moreover we are sure that $n > 2$ (Otherwise we could derive $x\ li\ y$). Let $l \in Lines(CS)$ with $Set(A) - \{x\} \subseteq l$. By P23 there is some $z \in l$ with $\{z, a_2\} = im[a_1]$. It should be clear that $z \in {}^\bullet a_{1F}$ ($z \in a_1{}^\bullet{}_F$ would imply $z\ co\ a_2$ contradicting $z, a_2 \in l$). Now we are prepared to apply ASS4a: As $z\ li\ y$ and $z, y \in co[a_0]$ this yields an $F|co[a_0]$-chain $B = (y, \ldots, z)$ (The reverse $F|co[a_0]$-chain $B = (z, \ldots, y)$ is excluded, because $z \in S$ and P35 and $a_0\ li\ a_1$ implies $a_1 \notin B$). Combining $A$ and $B$ gives an $F$-cycle $C = (z, \ldots, y, \ldots, z)$ contradicting the fact that $F$- is acyclic on $X$ which follows from the precondition that $CS$ is acyclic via P42. $\square$

It will be immediately proved that all doubly infinite $F$-chains are lines. So the class of acyclic concurrency structures has the advantage that lines can be directly determined once the flow relation is given.

**Corollary C10** Let $CS$ be acyclic and $(S,T,F) \in Nets(CS)$.
Then $A$ is an $\omega\omega$-infinite $F$-chain $\Rightarrow Set(A) \in Lines(CS)$. $\square$

**Proof** According to the previous proposition, $Set(A)$ is a clique of $\underline{li}_X$, which can be extended to a line $L$ with $Set(A) \subseteq L$. Assume $Set(A) \notin Lines(CS)$. Then $Set(A) \neq L$ and there is an $x \in L - Set(A)$. Clearly $Set(A) \subseteq li[x]$. Now choose some $y \in Set(A)$. As $x, y \in L$ D6j requires the existance of an $im$-chain $B = (x = x_0, x_1, \ldots, x_n = y)$ such that $Set(B) \subseteq L$. Let $i$ be the minimal index such that $x_i \notin Set(A)$. It follows that $i \geq 1$, $x_{i+1} \in Set(B)$ and $x_i\ im\ x_{i+1}$. Furthermore, $x_{i+1}$ has neighbors on $A$, say $u, v \in Set(A)$ with $u\ F\ x_{i+1}\ F\ v$ implying $u \neq v$ as $F$ is a consistent orientation. So there are three neighbors $\{u, v, x_i\} \subseteq im[x_{i+1}]$ all of them being contained in $L$, a contradiction with L7. $\square$

K-density, formulated in terms of the flow relation, yields for acyclic concurrency structures the following proposition.

**Proposition P45** Let $CS$ be acyclic $\wedge\ N = (S,T,F) \in Nets(CS)\ \wedge$
$v, x \in c \in Cuts(CS) \wedge$ $A, B$ be $F$-chains with $A = (u, \ldots, v) \wedge B = (x, \ldots, y)$.
Then $\forall C = (u, \ldots, y) : C$ is an $F$-chain $\Rightarrow Set(C) \cap c \neq \emptyset$. $\square$

**Proof** Assume $Set(C) \cap c = \emptyset$. Extending $C$ to an $\omega\omega$-infinite $F$-chain $D$ yields a line $Set(D) \in Lines(CS)$ according to C10. Now observe that $(F^{-1})^*_X[u] \cap c = \emptyset$. Otherwise choosing some $z \in (F^{-1}{}^*_X[u] \cap c$ we could construct an $F$-chain $(z, \ldots, u, \ldots, v)$ implying $z\ li\ v$ (by P44) which contradicts $z, v \in c$. Similarly we have $F^*_X[y] \cap c = \emptyset$. So we are sure that $Set(D) \cap c = \emptyset$, but by D6h we must have $Set(D) \cap c \neq \emptyset$. Contradiction. $\square$

Restricting concurrency theory to acyclic structures simplifies several proofs, as the transitivity of the order relation can be exploited. On the other hand, cyclic concurrency structures are finite, which is a simplification of a different kind. As a consequence it will be sometimes convenient to carry out a proof separately for these two cases.

### 5.11 Cone Intersection Property

In *Petri 1987* the cone intersection property was formulated for acyclic concurrency structures in the following way: Given two elements their past cones as well as their future cones of causality should have a nonempty intersection. More precisely, for every pair $x, y$ it is required that $F^+[x] \cap F^+[y] \neq \emptyset$ and $(F^+)^{-1}[x] \cap (F^+)^{-1}[y] \neq \emptyset$. Note that it is not essential, if the consistent orientation $F$ or its inverse is used to verify this property. As concurrency theory should not cover only acyclic structures but also cyclic ones, and the existence of a consistent orientation is no apriori postulate, it was recognized that a different formulation of this property is necessary.

The major reason for the cone intersection property is to ensure continuity for those processes described by concurrency structures. Unfortunately D-continuity has not been defined for cyclic structures, but in case of acyclic structures the relevance of the cone intersection property for D-continuity is known. Although the examination of D-continuity is beyond the scope of this treatment, it is believed that the axioms (together with our assumptions) are strong enough to ensure this important property. Furthermore, it should be mentioned, that for the reachability results and the link to elementary net systems presented in the subsequent sections the cone intersection property is not essential.

First we prove that the cone intersection property follows almost directly from our axiom of finite concurrency neighborhood (D6g) for the class of acyclic concurrency structures.

**Proposition P46** Let $CS$ be acyclic $\wedge\ x, y \in X$ with $x\ co\ y$.

a) $F_X^*[x] \cap F_X^*[y] \neq \emptyset$;
b) $(F_X^*)^{-1}[x] \cap (F_X^*)^{-1}[y] \neq \emptyset$.

□

**Proof** Using P42 we find that $F$ is acyclic. Assume our proposition is not satisfied, that is, there are $x, y \in X$ with $F_X^*[x] \cap F_X^*[y] = \emptyset$. Choose two arbitrary lines $l_x, l_y \in Lines(CS)$ with $x \in l_x$ and $y \in l_y$. As all lines $l \in Lines(CS)$ are infinite, we can apply P40 to find two infinite $F$-chains $A_x = (x, \ldots)$ and $A_y = (y, \ldots)$. By our assumption $Set(A_x) \cap Set(A_y) = \emptyset$ holds, and we claim that $Set(A_y) \subseteq co[x]$, which is infinite, contradicting D6g. To see that the claim holds notice that there cannot be any $z \in A_y$ with $x\ li\ z$, as C9 would require the existence of an $F$-chain $A_z = (x, \ldots, z)$ or $A_z = (z, \ldots, x)$. Both cases are not reconcilable with our assumption that $F_X^*[x] \cap F_X^*[y] = \emptyset$. □

As we have already mentioned, the axiom D6g seems to be very strong, and a concurrency theory yielding similar results with a weaker postulate might be desired. Then it is a natural question how to ensure the cone intersection property in terms of concurrency and causality without referring to the flow relation. One way to achieve this is to choose the following formulation, which was proposed in *Petri 1987* as a sufficient condition for the cone intersection property, as an axiom, although even this might be too strong.

**Axiom A15** [Sufficient Condition for Cone Intersection Property]
$\forall x, y \in X : x \; co \; y \Rightarrow$
$(\exists u, v, p, q : u \; li \; v \wedge u, v \in li[x] \cap li[y] \wedge x, y \in co[p] \cap co[q] \wedge$
$u \; co \; p \wedge v \; co \; q \wedge u \; li \; q \wedge v \; li \; p).$ □

This axiom states that every two concurrent elements $x$ and $y$ have some common elements $u$ and $v$ that are causally dependent with additional the requirement that $u$ affects $x$ as well as $y$ and $v$ is affected by $x$ and $y$ (the direction of this influence is essential here). The two elements $p$ and $q$ are used for technical reasons to ensure that $u$ and $v$ are located in opposite directions of time from the viewpoint of $x$ and $y$. The physical aim of the cone intersection property is to avoid partially frozen systems, that are systems where in some components the time evolution might stop although in other parts the dynamic continues without being affected. Loosely speaking, the cone intersection property states that $x$ and $y$ have been synchronized at $u$ and will resynchronize at $v$ (or vice versa).

First a simple but useful lemma is proposed exploiting the nice property of transitivity of the partial order associated with an acyclic concurrency structure.

**Lemma L22** Let $CS$ be acyclic.
Then $x \; co \; y \wedge x \; li \; z \wedge y \; li \; z \Rightarrow x \; F^+ \; z \wedge yF^+z \vee z \; F^+ \; y \wedge z \; F^+ \; x$. □

**Proof** As $F^+$ is transitive $x \; F^+ \; zF^+y$ would imply $x \; F^+ \; y$ which itself implies $x \; li \; y$ according to P44. Contradiction with $x \; co \; y$. Similarly, $y \; F^+ \; z \; F^+ \; x$ implies $y \; F^+ \; x$ and $y \; li \; x$. □

Successive application of this lemma and the axiom A15 gives the cone intersection property for acyclic concurrency structures, which is the same as P46, but the proof does not refer to D6g (at least not directly).

**Proposition P47** Let $CS$ be acyclic $\wedge \; x, y \in X \; \wedge \; x \; co \; y \; \wedge$ A15.

a) $F_X^*[x] \cap F_X^*[y] \neq \emptyset$;
b) $(F_X^*)^{-1}[x] \cap (F_X^*)^{-1}[y] \neq \emptyset$.

□

**Proof** We apply A15 in connection with P44. So there are $u, v, p, q$ satisfying the condition of A15. As $p \; li \; v$, we have either $p \; F^+ \; v$ or $v \; F^+ \; p$. We will only deal with $p \; F^+ \; v$, as the latter case is simply the $F$-reversal counterpart. In the following we make repetitive use of the fact L22. So $p \; F^+ \; v$ implies $u \; F^+ \; v$ due to $p \; co \; u$. With $u \; F^+ \; v$ we can only choose $u \; F^+ \; q$ due to $v \; co \; q$. With $q \; co \; y$ this implies $u \; F^+ \; y$. Similarly, $x \; F^+ \; v$ follows from $p \; F^+ \; v$ and $p \; co \; x$. Furthermore, we have $u \; F^+ \; x$ by $u \; F^+ \; q$ with $x \; co \; q$ and finally $y \; F^+ \; v$ by $p \; F^+ \; v$ and $p \; co \; y$. Altogether, we have necessarily $v \in F_X^*[x]$ and $v \in F_X^*[y]$ proving the first part and $u \in (F_X^*)^{-1}[x]$ and $u \in (F_X^*)^{-1}[y]$ which yields the second part of our proposition. □

The cone intersection property for cyclic concurrency structures has not been defined yet, although an immediate idea is to introduce it exactly as in the acyclic case (the definition is P46) with the help of some consistent orientation. If we proceed in this way, an acyclic concurrency structure generated by the unfolding of a cyclic one simply inherits the cone intersection property, such that the unfolding of a concurrency structure is again a concurrency structure. Strictly, we have not defined the concept of unfolding, but from the examples it should be clear what is meant.

With this general definition we could also examine if the cone intersection property is ensured by the list of axioms proposed for a concurrency structure. This establishes a direct relation to P42, where it was conjectured that it might be possible to prove: $CS$ is cyclic $\Rightarrow N$ is cyclic. Once we know that $N$ is cyclic, we have $F^+ = X \times X$ and $F^+[x] = X$ for every $x \in X$. Hence the cone intersection property would be satisfied.

## 5.12 Reachability

This section the question of dynamics in concurrency structures is addressed. First it is shown that conceiving cuts of local states (which will be called $S$-cuts) as markings of a net associated with the concurrency structure the common rule for the token game is completely equivalent to the propagation rules for concurrency. This means conceiving an $S$-cut as a marking and firing some transitions according to the rules of the token game we find a final marking which is always an $S$-cut. Furthermore it will be proved, and this will be more difficult and involves several conjectures proposed in the previous sections, that given two $S$-cuts interpreted as markings there is always a way applying the rules of the token game to reach one marking from the other. Note that, as the full and transitive reachability relation in nets is chosen as a basis for this treatment, transitions may be fired forward as well as backward.

As stated above the class of $S$-cuts contains exactly those cuts consisting of local states only.

**Definition D22** $SCuts(CS) := Cuts(CS) \cap \mathcal{P}(\mathcal{S})$. □

Before considering cuts we focus our attention on the behavior of concurrency cliques interpreted as markings, when the firing rule is applied. We find that given a concurrency clique and firing a transition that is enabled at the corresponding marking, we again obtain a clique of concurrency and, furthermore, the intermediate state, where the transition has just consumed the input tokens but not produced output yet, constitutes also a clique of concurrency. Of course, those tokens not touched by the selected transition are contained in each of these cliques. Actually, the following lemma is slightly more general, as it allows also events to be contained in these cliques and states the above implication also in its backward direction, which corresponds to the situation that the transition is fired backwards.

**Lemma L23** Let $(S, T, F) \in Nets(CS) \wedge t \in T \wedge$
$t \in c_2 \wedge c_1 = c_2 - \{t\} \cup {}^\bullet t_F \wedge c_3 = c_2 - \{t\} \cup t^\bullet{}_F$.
Then $c_1$ is a clique of $\underline{co}_X \Leftrightarrow c_2$ is a clique of $\underline{co}_X \Leftrightarrow c_3$ is a clique of $\underline{co}_X$. □

**Proof** Define $c := c_2 - \{t\}$. First we prove: $c_2$ is a clique of $\underline{co}_X \Rightarrow$ ($c_1$ is a clique of $\underline{co}_X$ $\wedge$ $c_3$ is a clique of $\underline{co}_X$). Assume $c_2$ is a clique $\underline{co}_X$. Then by P25 we have $\forall x \in c : \forall y \in im[t] : x \; co \; y$ such that $c_1$ and $c_3$ are cliques of $\underline{co}_X$.
Now we show: $c_1$ is a clique of $\underline{co}_X \Rightarrow c_2$ is a clique of $\underline{co}_X$. Assume $c_1$ is a clique of $\underline{co}_X$. Then for every $x \in c$ we can apply P28 as the condition $\forall y \in {}^\bullet t_F : x \; co \; y$ is satisfied (and P30 holds). So we infer $\forall x \in c : x \; co \; t$ and conclude that $c_2$ is a clique of $\underline{co}_X$. That $c_3$ is a clique of $\underline{co}_X \Rightarrow c_2$ is a clique of $\underline{co}_X$ is proved similar to the previous implications. □

The previous lemma can be extended to cuts in the following sense: Given a cut interpreted as a marking and firing a transition, that is enabled, again yields a cut. The only point, which is left to be proved, is that the maximality of a clique is preserved under the firing rule.

**Lemma L24** Let $(S, T, F) \in Nets(CS) \wedge t \in T \wedge$
$t \in c_2 \wedge c_1 = c_2 - \{t\} \cup {}^\bullet t_F \wedge c_3 = c_2 - \{t\} \cup t^\bullet{}_F$.
Then $c_1 \in Cuts(CS) \Leftrightarrow c_2 \in Cuts(CS) \Leftrightarrow c_3 \in Cuts(CS)$. □

**Proof** By L23 it is already clear that $c_1 \in Cuts(CS) \Rightarrow c_2$ is a clique of $\underline{co}_X$ and $c_2 \in Cuts(CS) \Rightarrow c_1$ is a clique of $\underline{co}_X$ such that all we have to prove the maximality of $c_2$ resp. $c_1$ concerning *co*.
We start with: $c_1 \in Cuts(CS) \Rightarrow c_2 \in Cuts(CS)$. Assume $c_2$ is not maximal with respect to *co*. Then $\exists c_2' \in Cuts(CS) : c_2 \subset c_2'$. With $c_1' := c_2' - \{t\} \cup {}^\bullet t_F$ applying L23 yields that $c_1'$ is a clique of $\underline{co}_X$. Obviously we have $c_1 \subset c_1'$ but this contradicts $c_1 \in Cuts(CS)$.
Similarly, we show: $c_2 \in Cuts(CS) \Rightarrow c_1 \in Cuts(CS)$. Again assume $\exists c_1' \in Cuts(CS) :$ $c_1 \subset c_1'$. Define $c_2' := c_1' - {}^\bullet t_F \cup \{t\}$. L23 implies that $c_2'$ is a clique of $\underline{co}_X$. But $c_2 \subset c_2'$ is not consistent with $c_2 \in Cuts(CS)$.
Exchanging ${}^\bullet t_F$ and $t^\bullet{}_F$ directly leads to the proof for $c_2 \in Cuts(CS) \Leftrightarrow c_3 \in Cuts(CS)$. □

By the help of this lemma one can construct an $S$-cut from every cut, that may contain some events, by firing the corresponding transitions.

**Lemma L25** Let $c \in Cuts(CS)$. Then $\exists c' \in SCuts(CS) : c \cap S \subseteq c'$. □

**Proof** Let $c_T = c \cap T$ and $c_S = c \cap S$. We construct the set $c' := c_S \cup \{s : \exists t \in c_T : s \in t^\bullet{}_X\}$ (${}^\bullet t_X$ would also do). With L24 it is clear that $c' \in Cuts(CS)$ and in particular $c' \in SCuts(CS)$. □

So we can easily prove that the class of $S$-cuts cannot be empty, which ensures that we can always find some initial marking, such that the formalism of elementary net systems will be applicable.

**Lemma L26** $SCuts(CS) \neq \emptyset$. □

**Proof** Choose some $x \in X$. Certainly there is a cut $c \in Cuts(CS)$ such that $x \in c$. Applying L25 to $c$ gives us a $c' \in SCuts(CS)$. □

For every transition we can find an $S$-cut, at which it is enabled, and a different $S$-cut, at which it is reverse enabled. Although this observation is trivial, it yields together with the following results a nice and desired property of elementary net systems.

**Remark R21** Let $t \in T$.

a) $\exists c \in SCuts(CS) : ({}^\bullet t_X \subseteq c)$;
b) $\exists c \in SCuts(CS) : (t^\bullet{}_X \subseteq c)$.

□

**Proof** Let $c_1 = {}^\bullet t_X$ (or $c_1 = t^\bullet{}_X$). Then $c_1$ is a clique of $\underline{co}_X$ and it can be extended to a cut $c_2 \in Cuts(CS)$ with $c_1 \subseteq c_2$. Applying L25 we can construct $c \in SCuts(CS)$ with $c \subseteq c_2$. □

The remainder of this section is concerned with the question if it is possible to establish the reachability relation between two arbitrary $S$-cuts conceived as the markings of a net associated with a concurrency structure. For this purpose we introduce two abbreviations:

$[s' \Uparrow c]_N$ and $[s' \Downarrow c]_N$. $s'$ is a place and $c$ is a set of places. Although these definitions are independent of the underlying concurrency structure, we will use this notation always with the assumption that $c$ is an $S$-cut, which should indicate the following meaning: $[s' \Uparrow c]_N$ is the set of net elements that can be obtained by looking into the future of $s'$, until we reach the specified cut $c$. Similarly, $[s' \Downarrow c]_N$ refers to the past of $s'$ up to $c$.

**Definition D23** Let $N = (S, T, F)$ be a net $\wedge c \subseteq S \wedge s' \in S$.

a) $[s' \Uparrow c]_N$ is the smallest set satisfying
   $s' \in [s' \Uparrow c]_N$ and $x \in [s' \Uparrow c]_N \wedge x \notin c \Rightarrow x^{\bullet}{}_F \subseteq [s' \Uparrow c]_N$.
b) $[s' \Downarrow c]_N$ is the smallest set satisfying
   $s' \in [s' \Downarrow c]_N$ and $x \in [s' \Downarrow c]_N \wedge x \notin c \Rightarrow {}^{\bullet}x_F \subseteq [s' \Downarrow c]_N$.

□

**Remark R22** $[s' \Uparrow c]_N$ and $[s' \Downarrow c]_N$ are well-defined. □

The following list of properties follows immediately from the definitions.

**Remark R23** Let $N = (S, T, F)$ be a net and $s' \in S \wedge c \subseteq S$.

a) $[s' \Uparrow c]_N = [s' \Downarrow c]_{N'}$ where $N' = (S, T, F^{-1})$;
b) $\forall t \in [s' \Uparrow c]_N \cap T : t^{\bullet}{}_F \subseteq [s' \Uparrow c]_N$;
c) $\forall t \in [s' \Downarrow c]_N \cap T : {}^{\bullet}t_F \subseteq [s' \Downarrow c]_N$;
d) $\forall s \in [s' \Uparrow c]_N \cap S - c : s'^{\bullet}{}_F \subseteq [s' \Uparrow c]_N$;
e) $\forall s \in [s' \Downarrow c]_N \cap S - c : {}^{\bullet}s'_F \subseteq [s' \Downarrow c]_N$;
f) $x \in [s' \Uparrow c]_N \Rightarrow x = s' \vee \exists y \in [s' \Uparrow c]_N - c : y \; F \; x$;
g) $x \in [s' \Downarrow c]_N \Rightarrow x = s' \vee \exists y \in [s' \Downarrow c]_N - c : x \; F \; y$;
h) $t \in T \wedge {}^{\bullet}t_F \subseteq c \Rightarrow t \notin [s' \Uparrow c]_N$;
i) $t \in T \wedge t^{\bullet}{}_F \subseteq c \Rightarrow t \notin [s' \Downarrow s]_N$;

□

Inductively we prove that for every $x \in [s' \Uparrow c]_N$ we find an $F$-chain from $s'$ to $x$ that is completely contained in $[s' \Uparrow c]_N$. Additionally, if $x$ is located on $c$, this is the only element on our chain that might meet $c$ (intuitively the chain ends as soon as it meets $c$). Otherwise our chain does not intersect with $c$, which is, loosely speaking, the situation that $x$ is located before $c$, or $c$ is located before $s'$ and cannot be reached by a finite chain, which might be the case for acyclic structures.

**Proposition P48** Let $N = (S, T, F) \in Nets(CS) \wedge s' \in S \wedge c \in Cuts(CS)$.

a) $\forall x \in [s' \Uparrow c]_N : \exists A : A = (s', \ldots, x)$ is an $F|[s' \Uparrow c]_N$-chain $\wedge$ $Set(A) \cap c = \{x\} \cap c$;
b) $\forall x \in [s' \Downarrow c]_N : \exists A : A = (x, \ldots, s')$ is an $F|[s' \Downarrow c]_N$-chain $\wedge$ $Set(A) \cap c = \{x\} \cap c$.

□

**Proof** Define $H(x) :\Leftrightarrow \exists A : A = (s', \ldots, x)$ is an $F|[s' \Uparrow c]_N$-chain $\wedge$ $Set(A) \cap c = \{x\} \cap c$. Obviously $H(s')$ holds if we simply choose $A = (s')$ such that $x = s'$. Assuming $H(x)$ holds and $x \notin c$ we prove $H(y)$ for all $y \in x^{\bullet}{}_F$. $H(x)$ yields an $F|[s' \Uparrow c]_N$-chain $A_x = (s', \ldots, x)$ with $Set(A_x) \cap c = \{x\} \cap c$. As $y \in [s' \Uparrow c]_N$ we append $y$ to $A$ which gives an $F|[s' \Uparrow c]_N$-chain $A_y = (s', \ldots, x, y)$. From $x \notin c$ we infer $Set(A_x) \cap c = \emptyset$ such that $Set(A_y) \cap c = \{y\} \cap c$. □

Furthermore, it will be show that, if $[s' \Uparrow c]_N$ is finite, it must contain some element $y$ of $c$ (intuitively the extension of $[s' \Uparrow c]_N$ into the future is restricted by $c$), which itself implies that an $F$-chain exists within $[s' \Uparrow c]_N$ from $s'$ to $y$. The last point again implies that $[s' \Uparrow c]_N$ is finite, because due to k-density there is no way to circumvent the cut $c$. Altogether this cyclic chain of implications establishes an equivalence between these statements enumerated in the next proposition.

**Proposition P49** Let $N = (S, T, F) \in Nets(CS) \wedge s' \in S \wedge c \in Cuts(CS)$.
Then the following statements are equivalent:

a) $[s' \Uparrow c]_N$ is finite;
b) $c \cap [s' \Uparrow c]_N \neq \emptyset$;
c) $\exists y \in c : \exists B : B = (s', \ldots, y)$ is an $F|[s' \Uparrow c]_N$-chain;
d) $\exists x \in c : \exists A : A = (s', \ldots, x)$ is an $F$-chain.

□

**Proof** $[s' \Uparrow c]_N$ is finite $\Rightarrow c \cap [s' \Uparrow c]_N \neq \emptyset$: Assuming $c \cap [s' \Uparrow c]_N = \emptyset$ we show that $[s' \Uparrow c]_N$ is infinite. If $CS$ is acyclic it is infinite by R18, and, as for every $x \in X$ we have $x^{\bullet}{}_F \neq \emptyset$, we get an infinite $[s' \Uparrow c]_N$. Otherwise $CS$ is cyclic and finite. Then we choose a line $l \in Lines(CS)$ with $s' \in l$ and by D6h there is some $z \in l \cap c$. As by P40 there is also an $F$-cycle $A = (s', \ldots, z, \ldots, s')$ with $Set(A) = l$, we certainly have $z \in [s' \Uparrow c]_N$. As $z \in c \cap [s' \Uparrow c]_N$ the implication holds.

$c \cap [s' \Uparrow c]_N \neq \emptyset \Rightarrow \exists x \in c : \exists A : A = (s', \ldots, x)$ is an $F|[s' \Uparrow c]_N$-chain: By P48a given $x \in c \cap [s' \Uparrow c]_N$ there is always an $F|[s' \Uparrow c]_N$-chain $A = (s', \ldots, x)$.

$\exists x \in c : \exists B : B = (s', \ldots, x)$ is an $F|[s' \Uparrow c]_N$-chain $\Rightarrow \exists y \in c : \exists A : A = (s' = a_0, a_1 \ldots, a_n = y)$ is an $F$-chain: Choose $A = B$.

$\exists y \in c : \exists B : B = (s', \ldots, y)$ is an $F$-chain $\Rightarrow [s' \Uparrow c]_N$ is finite: For cyclic $CS$ this is immediately clear by R18. So assume $CS$ is acyclic. Let $B = (s', \ldots, y)$ be an $F$-chain with $y \in c$ and assume $[s' \Uparrow c]_N$ is infinite. Then by P22 there must be an infinite $F|[s' \Uparrow c]_N$-chain $C = (s', \ldots)$ with $Set(C) \cap c = \emptyset$. Furthermore, we choose an infinite $F^{-1}$-chain $D = (s', \ldots)$. P44 suggests that there is a line $l_B \in Lines(CS)$ with $Set(D) \cup Set(B) \subseteq l_B$. Similarly, $l_C = Set(D) \cup Set(C)$ constitutes a line $l_C \in Lines(CS)$. As $l_B \cap c = \{y\}$ by P4 and there must be some $z$ with $l_C \cap c = \{z\}$, we conclude that $z \in Set(C)$. But this is a contradiction with $Set(C) \cap c = \emptyset$. □

Generally every statement concerning $[s' \Uparrow c]_N$ has a counterpart involving $[s' \Downarrow c]_N$, as it can easily seen by reversal of the flow relation.

**Proposition P50** Let $N = (S, T, F) \in Nets(CS) \wedge s' \in S \wedge c \in Cuts(CS)$.
Then the following statements are equivalent:

a) $[s' \Downarrow c]_N$ is finite;
b) $c \cap [s' \Downarrow c]_N \neq \emptyset$;
c) $\exists y \in c : \exists B : B = (y, \ldots, s')$ is an $F|[s' \Downarrow c]_N$-chain;
d) $\exists x \in c : \exists A : A = (x, \ldots, s')$ is an $F$-chain.

□

**Proof** Analogous to P49. □

Once we know that our concurrency structure is acyclic and that $[s' \Uparrow c]_N$ is finite, we can be sure that every transition-element occurring (directly) after $c$ cannot be contained in $[s' \Uparrow c]_N$. In other words, there is no chance starting at $s'$ to bypass the cut, if we follow the flow relation. Notice that this proposition cannot be generally satisfied in cyclic structures.

**Proposition P51** Let $CS$ be acyclic $\wedge\ N = (S, T, F) \in Nets(CS) \wedge$
$s' \in S \wedge c \in SCuts(CS)$.

a) $[s' \Uparrow c]_N$ is finite $\wedge s \in c \wedge t \in s^{\bullet}{}_F \Rightarrow t \notin [s' \Uparrow c]_N$;
b) $[s' \Downarrow c]_N$ is finite $\wedge s \in c \wedge t \in {}^{\bullet}s_F \Rightarrow t \notin [s' \Downarrow c]_N$.

□

**Proof** Let $[s' \Uparrow c]_N$ be finite, $s \in c$, $t \in s^{\bullet}{}_F$ and suppose $t \in [s' \Uparrow c]_N$. Then P48a suggests there is an $F|[s' \Uparrow c]_N$-chain $A = (s', \ldots, t)$ with $Set(A) \cap c = \emptyset$. Furthermore as there is some $x \in c \cap [s' \Uparrow c]_N$ by P49 we can apply P48a again which yields an $F|[s' \Uparrow c]_N$-chain $B = (s', \ldots, x)$ with $Set(B) \cap c = \{x\}$. Now observe that the preconditions of P45 are satisfied (we have two $F$-chains $B = (s', \ldots, x)$ and $D = (s, t)$ with $x, s \in c$). So we conclude that all $F$-chains $A = (s', \ldots, t)$ must meet $c$, that is, $Set(A) \cap c \neq \emptyset$. Contradiction. □

Given $s'$ and a cut $c$ there are two possibilities, which are not mutually exclusive in cyclic structures: $s'$ is located before $c$, which implies that $[s' \Uparrow c]_N$ is finite. Or $s'$ occurs after $c$, which means that $[s' \Downarrow c]_N$ is finite.

**Lemma L27** Let $N = (S, T, F) \in Nets(CS) \wedge s' \in S \wedge c \in SCuts(CS)$.
Then $[s' \Uparrow c]_N$ is finite $\vee$ $[s' \Downarrow c]_N$ is finite. □

**Proof** As $c \in Cuts(CS)$ there is some $x \in c$ with $s'\ \underline{li}_X\ x$. This implies by C9 that there is an $F$-chain $A = (s', \ldots, x)$ or $A = (x, \ldots, s')$. Applying P49 in the former case or P50 in the later case proves the lemma. □

We have seen that, if $[s' \Uparrow c]_N$ is finite, it must be restricted by $c$. As $c$ is a $S$-cut, we can find a transition with a postset covered by $c$. $c$, interpreted as a marking, enables this transition in backward direction. Similarly, if $[s' \Downarrow c]_N$ is finite, $c$ enables some transition in forward direction. In both cases this transition is also contained in $[s' \Uparrow c]_N$ resp. $[s' \Downarrow c]_N$. This proposition gives a first hint how the finiteness of either $[s' \Uparrow c]_N$ or $[s' \Downarrow c]_N$ may be exploited to decrease the distance between a cut $c$ and a part $s'$ of some other cut.

**Lemma L28** Let $N = (S, T, F) \in Nets(CS) \wedge s' \in S \wedge c \in SCuts(CS)$.

a) $[s' \Uparrow c]_N$ is finite $\Rightarrow \exists t \in T \cap [s' \Uparrow c]_N : t^{\bullet}{}_F \subseteq c$;
b) $[s' \Downarrow c]_N$ is finite $\Rightarrow \exists t \in T \cap [s' \Downarrow c]_N : {}^{\bullet}t_F \subseteq c$.

□

**Proof** Assume $[s' \Uparrow c]_N$ is finite and $\forall t \in [s' \Uparrow c]_N \cap T : \neg(t^{\bullet}{}_F \subseteq c)$. We conclude that $\forall t \in [s' \Uparrow c]_N \cap T : \exists s \in t^{\bullet}{}_F : s \notin c$ and with the additional fact that $\forall s \in [s' \Uparrow c]_N \cap S : s \notin c \Rightarrow s^{\bullet}{}_F \subseteq [s' \Uparrow c]_N$ we can construct an infinite $F|[s' \Uparrow c]_N$-chain $A = (s' = a_0, a_1, \ldots)$ with $a_i \notin c$ for all $i$. Now either $Set(A)$ is infinite, which contradicts our initial assumption that $[s' \Uparrow c]_N$ is finite, or $A$ is not acyclic, which means it contains an $F$-cycle $B$ with $Set(B) \subseteq Set(A)$ implying $Set(B) \cap c = \emptyset$. But this is not reconcilable with ASS3. A similar proof shows the second part of the lemma. □

With the previous lemma we can immediately derive a first reachability result: Imagine the situation, where we have an arbitrary cut $c$, which will be interpreted as a marking and some place $s'$, we want to reach (that is, it should be contained in some final marking $c'$ that is reachable from $c$) by firing transitions. That this is indeed possible is clear by the following argument: We know that $[s' \Uparrow c]_N$ or $[s' \Downarrow c]_N$ is finite. Repeated application of the previous lemma gives us a sequence of transitions to fire (which moves $c$ towards $s'$) successively reducing the cardinality of either $[s' \Uparrow c]_N$ of $[s' \Downarrow c]_N$. Finally there will be no transition left between $s'$ and the current marking, which means that $s'$ is actually reached. As only transitions contained in either $[s' \Uparrow c]_N$ or $[s' \Downarrow c]_N$ are used, we can even infer a stricter reachability $R_{N'}$ instead of $R_N$ where $N'$ is derived from $N$ by removing unnecessary transitions, that have not been fired in this procedure.

**Lemma L29** Let $N = (S, T, F) \in Nets(CS) \wedge s' \in S \wedge c \in SCuts(CS)$.

a) $[s' \Downarrow c]_N$ is finite $\Rightarrow \exists c' \in SCuts(CS) : s' \in c' \wedge c\ R_{N'}\ c'$
where $T' := T \cap [s' \Downarrow c]_N$, $F' := (F|S \cup T')$ and $N' := (S, T', F')$.
b) $[s' \Uparrow c]_N$ is finite $\Rightarrow \exists c' \in SCuts(CS) : s' \in c' \wedge c\ R_{N'}\ c'$
where $T' := T \cap [s' \Uparrow c]_N$, $F' := (F|S \cup T')$ and $N' := (S, T', F')$;

□

**Proof** We prove the first part of the lemma by induction. Assume $[s' \Downarrow c]_N$ is finite and define $H(n) :\Leftrightarrow (\mid [s' \Downarrow c]_N \cap T \mid = n \Rightarrow \exists c' \in SCuts(CS) : s' \in c' \wedge c\ R_{N'}\ c')$. $H(0)$ is satisfied as $\mid [s' \Downarrow c]_N \cap T \mid = 0$ implies $[s' \Downarrow c]_N = \{s'\}$ and $s' \in c$ such that we can simply choose $c' = c$. Assuming $H(n-1)$ holds for $n > 0$ we have to prove $H(n)$: L28 gives us a $t \in T'$ with ${}^\bullet t_F \subseteq c$. L24 yields a cut $c'' \in SCuts(CS)$, which is $c'' = (c - {}^\bullet t_F) \cup t^\bullet{}_F$ with $c\ R_N\ c''$, and as $t \in T'$ we also have $c\ R_{N'}\ c''$. Applying $H(n-1)$ to $c''$ (note that $\mid [s' \Downarrow c'']_N \cap T - \{t\} \mid = n - 1$) yields a $c' \in SCuts(CS)$ with $s' \in c'$ and $c''\ R_{N'}\ c'$. By transitivity of $R_{N'}$ we conclude $c\ R_{N'}\ c'$. □

**Remark R24** Let $N = (S, T, F)$ and $N' = (S, T', F')$ be nets with
$T' \subseteq T$ and $F' = F|(S \cup T')$.
Then $R_{N'} \subseteq R_N$. □

With the fact that additional transitions can never diminish the reachability and that at least one of the two sets $[s' \Downarrow c]_N$ or $[s' \Uparrow c]_N$ is finite we get a first important reachability result.

**Lemma L30** Let $N = (S, T, F) \in Nets(CS) \wedge s' \in S \wedge c \in SCuts(CS)$.
Then $\exists c' \in SCuts(CS) : s' \in c' \wedge c\ R_N\ c'$. □

**Proof** By L27 and L29 with the fact that $R'_N \subseteq R_N$ (R24). □

Given two cuts the distance between two cuts can be measured in terms of the cardinality of their intersection. If the overlapping is large we assume that we do not need much more effort to reach one from the other, such that the distance is short. If the intersection is empty, then we assume the two cuts are far away from each other.

From this point of view the preceding lemma provides a method to decrease the distance between two cuts, which are disjunct initially. Unfortunately this method does not generally succeed for two cuts, which have a non-empty intersection, as for finite structures both $[s' \Downarrow c]_N$ and $[s' \Uparrow c]_N$ are finite such that there is no global coordination determining on which way (for a cycle there are two major possibilities) these two cuts should approach each other. A solution to this problem is to start with the assumption that we have two cuts $c$ and $c'$ which intersect each other and to allow only those movements covered by the degree of freedom $CO(c \cap c')$. That is, speaking in terms of the token game, we simply fix those tokens in $c \cap c'$ and let $c$ and $c'$ approach each other using the same method we employed for our first reachability result.

**Scope S4** Let $N = (S, T, F) \in Nets(CS) \wedge c, c' \in SCuts(CS) \wedge$
$c \cap c' \neq \emptyset \wedge s' \in c' - c \wedge Z = CO(c \cap c')$.

We start with two overlapping $S$-cuts $c$ and $c'$ and the goal is to decrease the distance between $c$ and $c'$ (that is to increase their overlapping) interpreted as markings by applying the rules of the token game. As we want to achieve this within the degree of freedom $Z$ (that is we do not want to fire any transitions not contained in $Z$) we have to prove that $[s' \Downarrow c]_N$ or $[s' \Uparrow c]_N$ is finite and completely covered by $Z$.

**Scope S5** As $c \in Cuts(CS)$ given $s'$ there must be some $x \in c$ with $x \ \underline{li}_X \ s'$. ASS4a suggests that there is an $F|Z$-chain $A = (x, \ldots, s')$ or $A = (s', \ldots, x)$. For the following we assume that we are faced with the former case.

We will immediately show that in this case $[s' \Downarrow c]_N$ is necessarily contained in $Z$. The following lemma is even stronger than this claim but is not difficult to prove it by structural induction guided by the definition of $[s' \Downarrow c]_N$.

**Lemma L31** $\forall y \in [s' \Downarrow c]_N : \exists x \in c \cap \underline{li}_X[y] : \exists A : A = (x, \ldots, y)$ is an $F|Z$-chain. □

**Proof** We prove this by induction over $[s' \Downarrow c]_N$. For $y = s'$ the lemma holds by the assumption of 5.12. Assuming the lemma is proved for $y \notin c$ we show that it also holds for $y' \in {}^\bullet y_F$. So we have an $F|Z$-chain $A = (x, \ldots, y)$ for some $x \in c \cap \underline{li}_X[y]$. If $y \in S$ there is an unique $y'$ and $A$ has the form $A = (s, \ldots, y', y)$, which implies $y' \in Z$. Otherwise $y \in T$ and we exploit P25 to infer $y' \in Z$. As $c \in Cuts(CS)$ we find a $x' \in c$ with $x' \ li \ y'$. Applying ASS4a with the fact that $x', y' \in Z$ requires that either an $F|Z$-chain $A' = (x', \ldots, y')$ or an $F|Z$-chain $A' = (y', \ldots, x')$ exists. Let us consider the latter case first: If $y' \in T$ then $y \in S$ and $A = (x, \ldots, y', y)$ otherwise if $y' \in S$ then $y \in T$ and $A' = (y', y, \ldots, x')$. Combining $A$ and $A'$ we construct a further $F|Z$-chain $B = (x, \ldots, x')$ which yields a contradiction with ASS4b because $x \ co \ x'$. Hence we are sure that we can only have an $F|Z$-chain $A' = (x', \ldots, y')$. □

The anticipated claim immediately follows from this.

**Lemma L32** $[s' \Downarrow c]_N \subseteq Z$. □

**Proof** Immediately from L31. □

□ S5

That $[s' \Downarrow c]_N$ or $[s' \Uparrow c]_N$ (which can be proved similarly in case of $A = (s', \ldots, x)$) is contained in $Z$ is not enough. As reachability means that only a finite number of transitions may be fired we must ensure that one of these two sets is finite and has the additionally property to be covered $Z$.

**Lemma L33** $([s' \Downarrow c]_N$ is finite $\wedge [s' \Downarrow c]_N \subseteq Z) \vee ([s' \Uparrow c]_N$ is finite $\wedge [s' \Uparrow c]_N \subseteq Z)$. □

**Proof** With $A = (x, \ldots, s')$ in 5.12 we have found that $[s' \Downarrow c]_N \subseteq Z$. Applying P49 we additionally find that $[s' \Downarrow c]_N$ is finite. The other possibility is $A = (s', \ldots, x)$ which implies that $[s' \Uparrow c]_N$ is finite by P49 and we can proceed analogously to L31 to prove $[s' \Uparrow c]_N \subseteq Z$. $\square$

With the preceding lemma we derive the desired reachability result: Given $S$-cuts $c$ and $c'$ with non-empty intersection and an arbitrary $s'$ contained in $c'$ but not in $c$ we can find a further $S$-cut $c''$ that is reachable from $c$ and additionally covers $s'$. Those tokens contained in the intersection of $c$ and $c'$ are again contained in $c''$, that is, it is not necessary to move them.

**Lemma L34** $\exists c'' \in SCuts(CS) : s' \in c'' \wedge c \cap c' \subseteq c'' \wedge c\ R_N\ c''$. $\square$

**Proof** Assume we have a finite $[s' \Downarrow c]_N \subseteq Z$. Otherwise we must have a finite $[s' \Uparrow c]_N \subseteq Z$ according to L33 and an analogous proof succeeds. L29 gives us a $c'' \in SCuts(CS)$ with $s' \in c''$ and $c\ R_{N'}\ c''$, where $N' = (S, T', F')$, $T' = T \cap [s' \Downarrow c]_N$ and $F' = F|(S \cup T)$. Certainly $c\ R_{N'}\ c''$ implies $c\ R_N\ c''$, so it is only left to prove that $c \cap c' \subseteq c''$: For every $t \in [s' \Downarrow c]_N \cap T$ we have $im[t] \cap (c \cap c') = \emptyset$ by L32. This means that for all $m, m'$ with $m\ r_{N'}\ m'$ we have $(c \cap c') \subseteq m \Leftrightarrow (c \cap c') \subseteq m'$ (tokens in $c \cap c'$ are not touched) and lifiting this property to $R_{N'}$ we find with $c\ R_{N'}\ c''$ that $(c \cap c') \subseteq c$ (which is certainly true) implies $(c \cap c') \subseteq c''$. $\square$

$\square$ S5

Notice that the previous result indeed decreases the distance between $c$ and $c'$ as $c$ is carried over to $c''$ and the intersection of $c''$ and $c$ is strictly greater than the intersection of $c'$ and $c$. At least the element $s'$ is added to the set.

**Corollary C11** Let $N \in Nets(CS) \wedge c, c' \in SCuts(CS)$.
Then $\exists c'' \in SCuts(CS) : c \cap c' \subset c'' \cap c' \wedge c\ R_N\ c''$. $\square$

**Proof** We distinguish two cases: Either $c \cap c' = \emptyset$ or $c \cap c' \neq \emptyset$. In the first case we apply L30: We choose a $s' \in c'$ and the lemma gives us a $c'' \in SCuts(CS)$ with $c\ R\ c''$ and $s' \in c''$. Note that $c \cap c' = \emptyset \subset \{s'\} \subseteq c'' \cap c'$ trivially holds. In the latter case we have $c \cap c' \neq \emptyset$ and we apply L34: Choosing $s' \in c' - c$ yields a $c'' \in SCuts(CS)$ with $s' \in c''$, $c \cap c' \subseteq c''$ and $c\ R\ c''$. This implies $c \cap c' \subseteq c'' \cap c'$ and even $c \cap c' \subset c'' \cap c'$ because $s' \notin c \cap c'$ but $s' \in c'' \cap c'$. $\square$

Successive decrease of the distance between two cuts finally identifies them after a finite number of steps. Hence our main result concerning the reachability between two arbitrary $S$-cuts is established.

**Proposition P52** Let $N \in Nets(CS) \wedge c, c' \in SCuts(CS)$.
Then $c\ R_N\ c'$. $\square$

**Proof** To prove this by (reverse) induction over $\mid c \cap c' \mid$ (remember that $\mid c \cap c' \mid$ is finite by R3) we define $H(n) :\Leftrightarrow (\mid c \cap c' \mid = n \Rightarrow c\ R_N\ c')$. $H(\mid c' \mid)$ holds because $n = \mid c' \mid$ implies $c = c'$ and $c\ R_N\ c'$ holds trivially. Assuming $H(i)$ holds for all $i$ with $n < i \leq \mid c' \mid$ we prove $H(n)$: Given $c, c' \in SCuts(CS)$ we apply C11, which yields a $c'' \in SCuts(CS)$ satisfying $\mid c'' \cap c' \mid > \mid c \cap c' \mid = n$ and $c\ R_N\ c''$. So we can apply $H(i)$ for some $i = \mid c'' \cap c' \mid > n$ which proves $c''\ R_N\ c'$. By transitivity of $R_N$ we conclude $c\ R_N\ c'$. $\square$

## 5.13 Elementary Net Systems

The reachability results derived in the previous section indicate that elementary net systems are an adequate formalism yielding a concise representation of concurrency structures. We simply choose an arbitrary net associated with the concurrency structure and an arbitrary $S$-cut as initial case. Notice that the initial case has no further significance beyond the fact that it represents the case class of the elementary net systems.

**Scope S6** Let $(S, T, F) \in Nets(CS) \wedge c_0 \in SCuts(CS) \wedge NS = (S, T, F, c_0)$.

As all reachable cases are $S$-cuts in the concurrency structure, which have been required to be finite, we can identify the case class with the sequential case class, where only single transitions are fired to generate new cases.

**Proposition P53** $SeqCaseClass(NS) = CaseClass(NS)$. □

**Proof** It is sufficient to show $SR_N = R_N$. $SR_N \subseteq R_N$ is immediately clear. $R_N \subseteq SR_N$ follows from $r_N \subseteq SR_N$ by transitive closure. Given $c, c' \subseteq S$ with $c\ r_N\ c'$ there is an event $E$ with $c[E > c'$. As $c_0 \in SCuts(CS)$ we are sure that $c_0$ is finite (R3), and from P22 we infer that all cases $c \in CaseClass(CS)$ must be finite, too. So we can write $E$ as $E = \{t_0, \ldots, t_n\}$ and we find a finite sequence $(c = c_0, \ldots, c_{n+1} = c')$ with $c_i[t_i > c_{i+1}$ for $i \in [0, n]$ proving $c\ SR_N\ c'$. □

Stated in a different form, the reachability result of the previous section simply corresponds to the identification of the case class of the elementary net systems with the class of $S$-cuts of the underlying concurrency structure.

**Proposition P54** $CaseClass(NS) = SCuts(CS)$. □

**Proof** Let $NS = (S, T, F, c_0)$ with $c_0 \in SCuts(CS)$.
$CaseClass(NS) \subseteq SCuts(CS)$: By P53 we can equivalently prove $SeqCaseClass(NS) \subseteq SCuts(CS)$. We have $c_0 \in SCuts(CS)$ and according to L24 we cannot leave the class of $SCuts(CS)$ by firing single transitions forward or backward.
$SCuts(CS) \subseteq CaseClass(NS)$: It is sufficient to show that for every $c \in SCuts(CS)$ we have $c_0\ R_N\ c$ which can be directly derived using P52 and the fact that $c_0 \in SCuts(CS)$. □

There is actually no information lost in carrying out the step from concurrency structures to elementary net systems, because the concurrency relation between local states can always be reconstructed from the the case class. The full concurrency relation can then be easily derived using the propagation rules (in particular P28), and finally the relation of causality emerges as the irreflexive complement.

**Remark R25** Let $s, s' \in S$.
Then $s\ \underline{co}_X s' \Leftrightarrow \exists c \in CaseClass(NS) : s, s' \in c$. □

The nice and desirable property that the elementary net systems is safe as well as secure is a direct consequence of the fact that immediate temporal predecessors are always causally related to the immediate temporal successors. The property of being safe can be seen as a natural property of physical systems, where every aspect of causality should be covered by the causality relation $li$ and should not emerge from contacts one the level of the token game (that is, a cut should not have a causal influence on itself). Security can be interpreted in the following way: Transitions resp. events should only wait for their preconditions to be satisfied and become activated without care of their postconditions, which are anyway located in the future and therefore not accessible.

**Proposition P55** $NS$ is safe and secure. □

**Proof** To violate the condition that $NS$ is secure it is necessary to have a transition $t \in T$ and a case $c \in CaseClass(NS)$ with $s, s' \in c$ such that $s\ F\ t \wedge t\ F\ s'$. But this implies $s\ li\ s'$ (by R12) which is impossible since $c \in SCuts(CS)$ according to P54. This proves that $NS$ is secure which also implies the weaker property that $NS$ is safe. □

That every transition's preconditions (and postconditions) are covered by some $S$-cut shows, as every $S$-cut is reachable, that every transition is involved in the dynamics of our elementary net systems and may eventually fire. This implies that no transition is superfluous, which is also true for every place, as every single local state may be extended to a (reachable) $S$-cut.

**Proposition P56** $NS$ is proper. □

**Proof** From the fact that $N$ is connected we can derive $T = ProperT(NS) \Rightarrow S = ProperS(NS)$. So it is sufficient to prove $T = ProperT(NS)$. For every $t \in T$ we know that ${}^\bullet T_F$ is a clique of $\underline{co}_X$ and can be extended to a cuts $c \in SCuts(CS)$ with ${}^\bullet t_F \subseteq c$. By P54 we have $c \in CaseClass(CS)$, such that, as $NS$ is safe, there is some $c' \in CaseClass(CS)$ with $c[t > c'$ proving that $t \in ProperT(NS)$. □

□ S6

□ S6

The possibility to describe a concurrency structure by an elementary net systems yields a simple and compact notation and facilitates reasoning about the dynamics, as the usual token game can be applied. This is demonstrated by the following two examples in Fig. 10, which are elementary net systems that correspond to the concurrency structures of Fig. 1.a and Fig. 2.a. Of course the inverse nets or other cuts as initial markings would also be possible.

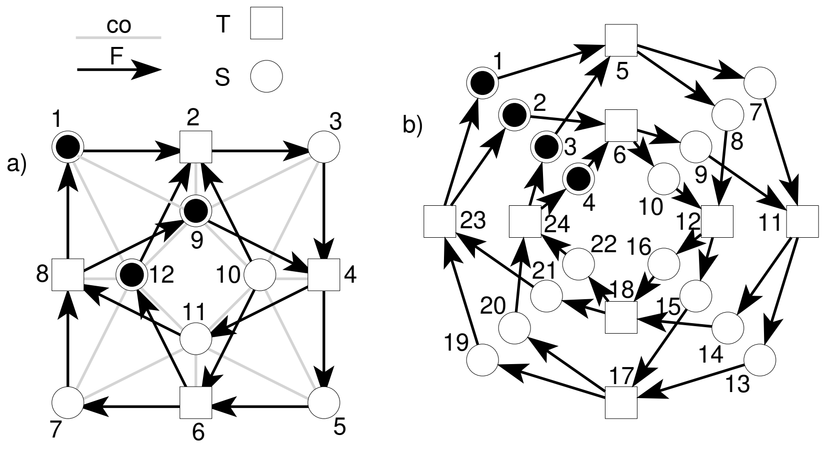


Figure 10: Concurrency structures as elementary net systems

# 6 Conclusion and Open Questions

In this treatment we started out from the basic notions of causality and concurrency as binary relations one some uniform set. In our standard interpretation concurrency structures are intended for the representation of acyclic as well as cyclic processes in time-space excluding the existence of state-space (that would be spanned by alternatives). Several mainly physically motivated postulates have been introduced as axioms that a concurrency structure should satisfy in order to conform to physical experience. After the clarification of elementary properties of concurrency and causality following from these axioms a considerable amount of effort has been spent to build a bridge from concurrency theory to the formalism of nets. That the suggested correspondence between a concurrency structure and a set of two nets of which one is the inverse of the other is reasonable is justified finally by the major result that the class of $S$-cuts exactly coincides with the case class of an elementary net systems based upon one of these nets.

In the beginning it was mentioned that our set of concurrency axioms is a slight modification of the original theory. At first, the property of coherence on lines (D6f), which replaces the original axiom of $im$-coherence $im_X^* = X \times X$, is necessary to show that the class of $S$-cuts is connected in terms of the reachability-relation. If there is a line, where between two elements there is an infinite number of further elements (as it was shown to be possible in *Best und Merceron 1985*), there is no finite firing sequence (on the level of elementary net systems), that could establish a link between two $S$-cuts, if each of them contains one of these elements. The second modification of the axioms is a restriction to structures with a finite concurrency neighborhood ($\forall x \in X : co[x]$ is finite), which seems to be even stronger than the requirement for finite cuts. None of these restrictions have be present in the original concurrency theory and it is not clear if they are really necessary. Formally this additional axiom helps us to derive the existence of exactly two classes of concurrency structures, namely cyclic and acyclic ones. Furthermore the axiom leads to the cone-intersection-property for acyclic structures, which is difficult to formalize without an underlying partial order.

A first hint that concurrency theory can be formulated without assuming an underlying partial order was already given in *Petri 1980a.* To our knowledge a consequent examination of this approach does not exist. The idea that the introduction of a consistent orientation on a concurrency structure yields exactly two nets was stated in *Petri 1988b* without proofs and a connection between concurrency theory and elementary net systems, in particular that the case-class might coincide with the class of $S$-cuts, has already been conjectured in *Petri 1980a.*

Unfortunately, we have not succeeded in proving all of these properties mentioned above directly from the axioms that have been chosen as a starting point. Indeed some assumptions have been necessary to derive the final results. Hopefully, these assumptions can be proved in near future directly from the proposed axioms (maybe with additional but more elementary assumptions), but, if this is not the case, one has to pose the question if these axioms are really strong enough. For the moment we have to conceive these assumptions as additional axioms, which may be not independent of the other ones. It is believed that some of these assumptions, in paticular ASS3 and ASS4 as not so restricting as it might seem. They might be satisfied for cyclic models derived from acyclic ones, by a folding which is large enough in the temporal direction. Apart from investigating these assumptions there are several further interesting questions worth to study.

An immediate question emerging from the preceding section is whether the initial case in elementary net systems is really necessary to reconstruct the underlying concurrency structure. It might be possible to prove that under certain conditions (e.g. the requirement for proper and secure elementary net systems) there is only one possible case class for a given net, which could be called the natural case class of that net.

A second interesting aspect of concurrency theory which should be addressed in a more general framework is the notion of cyclic, partial orders which could be defined in such a way that not a full but only a modest transitivity is required which just prevents the order relation from getting meaningless (that is the case if every element is before every other one). Once an appropriate definition of cyclic, partial orders is found it might give a more convenient formal basis for concurrency theory. In fact an alternative and interesting approach for dealing with cyclic structures has been proposed in *Petri 1980a* on the basis of so-called separation quadruples which are simply unordered pairs of unordered pairs $\{\{u, v\}, \{x, y\}\}$ such that $u$ and $v$ separate $x$ and $y$ on some line. The set of all separation quadruples can be used as an alternative representation of concurrency structures. Certainly it provides a unified frame to cope with cyclic as well as acyclic structures, but as this set is quite large, it is not clear, if it really facilitates reasoning about concurrency theory.

An important and probably the most ingenious concept that is related to concurrency theory is the notion of D-continuity that is very different from usual approach to continuity. A first goal might be to give an adequate definition of D-continuity, that is appropriate for cyclic as well as acyclic structures (a formalism of generalized orders will certainly helpful here). A second step is to examine if the axioms of concurrency theory really imply D-continuity or if some further requirements are necessary. But not only horizontal continuity (that is continuity within one level of abstraction) is an import issue. The vertical continuity (which is a topological continuity between different views) is also worth to investigate. In particular the relation between these apparently different forms of continuity would be interesting.

A further direction of future investigation is certainly driven by the demand for the application of concurrency theory to real-world problems. So far it is even not clear how to describe elementary experiments of classical mechanics using concurrency theory or if it is necessary to extend the theory in some way to achieve this. Certainly the connection between D-continuity and continuous movements has to be exploited here. Related to these questions is obviously the necessity to represent analogous quantities like position, velocity, acceleration and their measurement. Is it really possible to provide a solution within concurrency theory?

We have seen that the infinite two-dimensional grid is not K-dense. Interestingly, folding this grid to a torus in different ways yields the special subclass of finite and regular concurrency structures called cycloids. Cycloids have many nice mathematical properties and are conjectured to serve as a basis to solve technical safety and security problems in repetitive dynamical systems. The most astonishing feature of the class of cycliods is probably that cycloids can be transformed to other cycloids with the help of the Lorentz-transformation known from special relativity theory. The number of events and local states turn out to be Lorentz-invariants. So the Lorentz-transformation corresponds to a retiming-transformation and might have applications in the field of data flow systems.

A final and this is probably the most difficult demand is to include alternatives into the theory to contribute to the needs of information processing. So far concurrency theory only covers the aspect of synchronization, but for the storage of concrete bits and their transformation we need a means to express the dimension of state-space. Once information processing is captured by the theory, it might help to cope with general physics, if this is viewed as an informational process. A formally elegant way to introduce state-space is to define a relation *al* (for mutually exclusive alternatives) which might share up to a certain degree many properties of *co*. At some level then the theory must mirror the fundamental difference between concurrency and alternative. Certainly there will be several physical restrictions to be formulated as axioms (We must look out for physical laws, which remain invariant, if we change the level of abstraction.), and some of them might be motivated from the field of quantum mechanics. The wave/particle-duality of quantum-mechanics, for instance, might indicate that different possibilities for future temporal evolution can be taken concurrently (for waves) as well as in strict alternative fashion (for particles). This observation could motivate the possibility of non-empty intersection between *al* and *co*. The general aim might be to develop a deterministic and reversible theory (for closed systems) such that on the level of nets, which are already equipped with a means to express alternatives (by branched places), a deterministic and reversible dynamics (that is all forward and backward alternatives are solved) emerges with the usual token game.

As a theory including alternatives and concurrency simultaneously will be quite complex, a first approach might be to consider structures with causality and alternatives but excluding concurrency. Certainly this is not sufficient for information processing purposes, as we have no concurrent flows of information, which could interact with each other (as we know it from the theory of information-flow-graphs). But nevertheless we can pose several questions: Does this theory of alternatives simply correspond to a dualization of concurrency theory and their associated nets? What is the meaning of K-density in this context? What about the Lorentz-transformation mentioned above? Is there some counterpart for it in this theory of alternatives, and has it any physical significance?

## 7 Acknowledgements and Recent Works

This work is a revised version of the author's "Studienarbeit" (*Stehr 1993*) finished in December 1993 at the Computer Science Department of the University of Hamburg. The supervisor was Prof. Dr. Jozef Gruska (Slovak Academy of Science) a major focus of his interests being the relation between physics and computer science. He was open for numerous fruitful discussions and contributed to this work with competent advice. Some modifications introduced in this revised version are due to Olaf Kummer, who wrote a "Diplomarbeit" (*Kummer 1996*) on this subject, where many of the ideas have been further elaborated. In particular a proof of ASS1 has been given there. All of the other assumptions ASS2, ASS3 and ASS4 have been found to be independent of the axioms. It has been furthermore shown that ASS3 and ASS4 can be deduced from appearently more elementary assumptions about concurrency and causality. Also the proofs based on ASS2 has been optimized. The difficulties with cyclic concurrency structures initiated the

axiomization and investigation of (partial) cyclic orders in *Stehr 1996*, which are of similar mathematical generality as partial orders. Concurrency theory on the basis of cyclic orders is certainly possible, but has not been studied so far.

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# A Notation and Basic Definitions

We will use first order predicate logic with the usual symbols $\neg$,$\wedge$,$\vee$,$\Rightarrow$,$\Leftrightarrow$ and $\exists$, $\forall$. For better readability we will often use natural language to formulate logical statements. In particular relations and functions are frequently notated by special symbols or in natural language to remember the meaning of their arguments. In any case the notation should be precise enough to avoid loss of formal accuracy.

## Sets

We will assume the usal axioms of elementary set theory based on the binary element-predicate $\in$ only which postulate the existance of certain sets. If $H(x)$ is a predicate with $x$ as free variable then sets will be constructed with the notation $\{x : H(x)\}$ applying the comprehension axiom. Furthermore we will use the following conventional abbreviations:

**Definition D24**

a) $A = B :\Leftrightarrow \forall a : (a \in A \Leftrightarrow a \in B)$;
b) $A \cup B := \{x : x \in A \vee x \in B\}$;
c) $A \cap B := \{x : x \in A \wedge x \in B\}$;
d) $\overline{A} := \{a : a \notin A\}$;
e) $B - A := B \cap \overline{A}$;
f) $A \subseteq B :\Leftrightarrow A \cup B = B$;
g) $A \subset B :\Leftrightarrow A \subseteq B \wedge A \neq B$;
h) $\mathcal{P}(\mathcal{A}) := \{B : B \subseteq A\}$;
i) $\overline{A}_X := X - A$;
j) $\bigcup A := \{x : \exists a \in A : x \in a\}$;
k) $\bigcap A := \{x : \forall a \in A : x \in a\}$.

□

The usal order and operations on natural numbers and integers are assumed.

**Definition D25**

a) $\mathbb{N} := \{0, 1, 2, \ldots\}$;
b) $\mathbb{N}[a, b] := \{x : a \leq x \wedge x \leq b\}$.
c) $\mathbb{Z} := \{\ldots, 2, -1, 0, 1, 2, \ldots\}$;

□

## Tuples and Relations

*Tuples* are defined in the conventional way. They are denoted by $(a_0, a_1, \ldots, a_n)$. A *relation* is a subset of a cartesian product of sets:

**Definition D26** $A_0 \times A_1 \times A_2 \times \ldots \times A_n :=$
$\{(a_0, a_1, a_2, \ldots, a_n) : a_0 \in A_0 \wedge a_1 \in A_1 \wedge a_2 \in A_2 \wedge \ldots \wedge a_n \in A_n\}$. □

In particular we are interested in binary relations $S, R \subseteq X \times X$. Hence we use the following abbreviations:

**Definition D27**

a) $a \; R \; b :\Leftrightarrow (a, b) \in R$;
b) $Id_A := \{(a, a) : a \in A\}$;
c) $R^{-1} := \{(b, a) : aRb\}$;
d) $\overline{R}_X := X \times X - R$
e) $R[A] := \{b : \exists a \in A : aRb\}$;
f) $Dom(R) := \{a : \exists b : aRb\}$ *(domain of* $R$*)*;
g) $Ran(R) := \{b : \exists a : aRb\}$ *(range of* $R$*)*;
h) $Base(R) := Dom(R) \cup Ran(R)$ *(base of* $R$*)*;
i) $\underline{R}_X := R \cup id_X$;
j) $\widehat{R}_X := R \cup R^{-1} \cup Id_X$;
k) $R \odot S := \{(a, c) : \exists b : (aRb \wedge bSc)\}$;
l) $R \circ S := S \odot R$;
m) $R^1 := R$; $R^{n+1} := R^n \odot R$;
n) $R^+ := \bigcup_{n \in \mathbb{N}} R^{n+1}$;
o) $R^*_X := R^+ \cup Id_X$;
p) $\updownarrow R := R \cup R^{-1}$;
q) $R|X := R \cap (X \times X)$;
r) $a\tilde{R}_X b :\Leftrightarrow \underline{R}_X[a] = \underline{R}_X[b]$.

□

**Definition D28**

a) $R$ *is reflexive on* $X :\Leftrightarrow Id_X \subseteq R$;
b) $R$ *is irreflexive on* $X :\Leftrightarrow Id_X \cap R = \emptyset$;
c) $R$ *is symmetric* $:\Leftrightarrow R = R^{-1}$;
d) $R$ *is asymmetric* $:\Leftrightarrow R \cap R^{-1} = \emptyset$;
e) $R$ *is antisymmetric on* $X :\Leftrightarrow R \cap R^{-1} \subseteq Id_X$;
f) $R$ *is transitive* $:\Leftrightarrow R \circ R \subseteq R$;
g) $R$ *is a similarity on* $X :\Leftrightarrow R|X$ is reflexive on $X$ and symmetric;
h) $R$ *is an equivalence on* $X :\Leftrightarrow R|X$ is reflexive on $X$, symmetric and transitive.

□

**Definition D29**

a) $R$ *is cyclic on* $X :\Leftrightarrow R^+ = X \times X$;
b) $R$ *is acyclic on* $X :\Leftrightarrow R^+ \cap id_X = \emptyset$.

□

**Definition D30**

a) $(X, R)$ *is a poset* $:\Leftrightarrow R$ is transitive, reflexive on $X$ and antisymmetric;
b) $(X, R)$ *is a strict poset* $:\Leftrightarrow R$ is transitive, irreflexive on $X$ and asymmetric.

□

**Definition D31** Let $(X, R)$ be a poset.

a) $(X, R)$ *is total* $:\Leftrightarrow X \times X \subseteq R \cup R^{-1}$.
b) $Y$ *is a* chain *of* $(X, R)$ $:\Leftrightarrow Y \subseteq X$ and $(Y, R)$ is total.
c) $Chains(X, R) := \{(Y, R) : Y$ is a chain of $(X, R)$ $\}$.

$\square$

**Definition D32** Let $(X, R)$ be a poset and $Y \subseteq X$.

a) $Max(R, Y) := \{y : y \in Y \wedge \neg\exists z \in Y : (y\ R\ z \wedge z \neq y)$;
b) $Min(R, Y) := \{y : y \in Y \wedge \neg\exists z \in Y : (z\ R\ y \wedge z \neq y)$;
c) $Max(R) := Max(R, Base(R))$;
d) $Min(R) := Min(R, Base(R))$;

$\square$

**Definition D33** Let $(X, R)$ be a poset and $Y \subseteq X$.

a) $UpperBounds(R, Y) := \{x : \forall y \in Y : y\ R\ x\}$;
b) $LowerBounds(R, Y) := \{x : \forall y \in Y : x\ R\ y\}$;
c) $Sup(R, Y) := Min(R, UpperBounds(R, Y))$;
d) $Inf(R, Y) := Max(R, LowerBounds(R, Y))$.

$\square$

**Axiom A16** [Zorn's Lemma] Let $(X, R)$ be a poset.
$(\forall Y \in Chains(X, R) : Sup(R, Y) \neq \emptyset) \Rightarrow (Max(R) \neq \emptyset)$. $\square$

**Definition D34** Let $R \subseteq X \times X$ be a similarity and $S \subseteq X$.

a) $S$ *is a clique of* $R$ $:\Leftrightarrow S \times S \subseteq R$;
b) $Cliques(R) := \{S : S$ is a clique of $R$ $\}$;
c) $S$ *is a ken of* $R$ $:\Leftrightarrow S \in Max(\subseteq, Cliques(R))$;
d) $Kens(R) := \{S : S$ is a ken of $R$ $\}$.

$\square$

**Definition D35** Let $R$ be an equivalence on $X$.

a) $[x]_R := \{x' : x'Rx\}$ (*the equivalence class of $x$ with respect to $R$*);
b) $X/R := \{[x]_R : x \in X\}$ ($X$ *modulo* $R$).

$\square$

**Definition D36** Let $R \subseteq X \times X$ be an equivalence on $X$ and $S \subseteq X \times X$.
$S/R := \{([a]_R, [b]_R) : a\ S\ b\}$ ($S$ *modulo* $R$). $\square$

## Functions

**Definition D37** $F : A \to B :\Leftrightarrow$
$F \subseteq A \times B$ and $\forall (a, b) \in F : \forall (c, d) \in F : (a = c \Rightarrow b = d)$
($F$ *is a (partial) function from* $A$ *to* $B$). $\square$

**Definition D38** $F$ *is a function* $:\Leftrightarrow \exists A, B : F : A \to B$. $\square$

**Definition D39** Let $F$ be a function.

a) $y = F(x) :\Leftrightarrow (x, y) \in F$;
b) $F^{-1}(y) := \{x : y = f(x)\}$.

$\square$

**Definition D40** Let $F$ be a function.

a) *$F$ is total on $A$* $:\Leftrightarrow Dom(F) = A$;
b) *$F$ is surjective on $B$* $:\Leftrightarrow Ran(F) = B$;
c) *$F$ is injective* $:\Leftrightarrow function(F) \wedge function(F^{-1})$;
d) *$F$ is bijective on $B$* $:\Leftrightarrow$ $F$ is surjective on $B$ and injective.

$\square$

## Sequences

**Definition D41** Let $F$ be a function.

a) *$F$ is a finite sequence* $:\Leftrightarrow Dom(F) \subseteq \mathbb{N}[0, n]$ for some $n$;
b) *$F$ is a $\omega$-sequence* $:\Leftrightarrow Dom(F) = \mathbb{N}$;
c) *$F$ is a $\omega\omega$-sequence* $:\Leftrightarrow Dom(F) = \mathbb{Z}$;
d) *$F$ is a sequence* $:\Leftrightarrow$ $F$ is a finite, $\omega$-, or $\omega\omega$-sequence.

$\square$

We will denote a sequence $F$ by $(x_i)_{i \in I}$ where $I = Dom(F)$ and $x_i = F(i)$. A finite sequence $F$ may also be notated as $(x_0, x_1, \ldots, x_n)$, a $\omega$-sequence as $(x_0, x_1, \ldots)$ and a $\omega\omega$-sequence as $(\ldots, x_{-1}, x_0, x_1, \ldots)$.

**Definition D42** Let $F = (x_i)_{i \in I}$ be a sequence.

a) $Index(F) := Dom(F)$ *the index set of $F$*;
b) $Set(F) := Ran(F)$ *the range of $F$*;
c) $x \in F :\Leftrightarrow x \in Set(F)$ *$x$ is contained in $F$*.
d) $\mid F \mid = \mid I \mid$ *the size of $F$*;
e) $First(F) = F(z)$ if $\{z\} = Min(\leq, I))$ *the first element of $F$*;
f) $Last(F) = F(z)$ if $\{z\} = Max(\leq, I))$ *the last element of $F$*;

$\square$

**Definition D43** Let $F$ be a sequence.

a) *$F$ is a $R$-chain* $:\Leftrightarrow \forall i, j \in Index(F) : i + 1 = j \Rightarrow F(i)\ R\ F(j)$;
b) *$F$ is a $R$-cycle* $:\Leftrightarrow$ $F$ is a $R$-chain and $Last(F) = First(F)$.

$\square$

**Definition D44** Let $F$ be an $R$-chain.

a) $F$ is cyclic $:\Leftrightarrow Last(F) = First(F)$;
b) $F$ is acyclic $:\Leftrightarrow \forall i, j \in Index(F) : i \neq j \Rightarrow F(i) \neq F(j)$.

$\square$

**Definition D45** Let $R, S \subseteq X \times X$ and $F$ be a $R$-chain.
$S|F = \{(F_i, F_j) : j = i + 1 \wedge F_i\ S\ F_j\}$. $\square$

## Nets

**Definition D46** $(S,T,F)$ *is a net* $:\Leftrightarrow$
$S \cap T = \emptyset \ \wedge F \subseteq (S \times T) \cup (T \times S)$. $\Box$

**Definition D47** Let $N = (S,T,F)$ be a net.

a) $S_N := S$ (*the set of S-elements or places*);
b) $T_N := N$ (*the set of T-elements or transitions*);
c) $F_N := F$ (*the flow relation*);
d) $X_N := S \cup T$ (*the set of net elements*).

$\Box$

**Definition D48** Let $N = (S,T,F)$ be a net and $Y \subseteq X_N$.

a) $Y^{\bullet}{}_F := F[Y]$ (*the postset of Y with respect to F*);
b) ${}^{\bullet}Y_F := F^{-1}[Y]$ (*the preset of Y with respect to F*).

$\Box$

**Definition D49** Let $N = (S,T,F)$ be a net and $X = X_N$.

a) $N$ *is primitive* $:\Leftrightarrow \forall x \in X : {}^{\bullet}x_F \cup x^{\bullet}{}_F \neq \emptyset$;
b) $N$ *is pure* $:\Leftrightarrow \forall x \in X : ({}^{\bullet}x_F \cap x^{\bullet}{}_F = \emptyset)$;
c) $N$ *is simple* $:\Leftrightarrow \forall x, y \in X : ({}^{\bullet}x_F = {}^{\bullet}y_F \wedge x^{\bullet}{}_F = y^{\bullet}{}_F \Rightarrow x = y)$;
d) $N$ *is connected* $:\Leftrightarrow (F \cup F^{-1})^{+} = X \times X$;
e) $N$ *is cyclic* $:\Leftrightarrow F^{+} = X \times X$;
f) $N$ *is acyclic* $:\Leftrightarrow F^{+} \cap id_X = \emptyset$;

$\Box$

**Definition D50** Let $N = (S,T,F)$ and $N' = (S',T',F')$ be nets.

a) $N'$ *is a subnet of* $N$ $:\Leftrightarrow$
$S' \subseteq S \wedge T' \subseteq T \ \wedge$
$F' = F \cap ((S' \times T') \cup (T' \times S'))$;

$\Box$

**Definition D51** Let $N = (S,T,F)$ be a net.

a) $t$ *and* $t'$ *are independent in* $N$ $:\Leftrightarrow$
$({}^{\bullet}t_F \cup t^{\bullet}{}_F) \cap ({}^{\bullet}t'_F \cup t'^{\bullet}{}_F)$.

$\Box$

**Definition D52** Let $N = (S,T,F)$ be a net.

a) $M$ *is a marking in* $N$ $:\Leftrightarrow M \subseteq S$;
b) $E$ *is an event in* $N$ $:\Leftrightarrow E \subseteq T$ and $E \neq \emptyset$;
c) $(M[E > M')_N$ $:\Leftrightarrow$
$M, M' \subseteq S$ and $E \subseteq T$ and
$\forall t, t' \in E : (t \neq t' \Rightarrow t^{\bullet}{}_F \cap t'^{\bullet}{}_F = \emptyset \wedge {}^{\bullet}t_F \cap {}^{\bullet}t'_F = \emptyset) \ \wedge$
$(\bigcup t^{\bullet}{}_F : t \in E = M' - M \wedge \bigcup {}^{\bullet}t_F : t \in E = M - M')$
($M'$ *is reachable from* $M$ *by the step* $E$ *in* $N$);

d) *$E$ is a step in $N$* $:\Leftrightarrow \exists M, M' : M[E > M'$;
e) $Steps(N) := \{E : E$ is a step in $N\}$;
f) $M \; r_N \; M' :\Leftrightarrow \exists E : M[E > M'$ (*$M'$ is reachable from $M$ in one step in $N$*);
g) $M \; R_N M' :\Leftrightarrow M \; (r_N \cup {r_N}^{-1})^*_{\mathcal{P}(\mathcal{S})} \; M'$ (*$M'$ is reachable from $M$ in $N$*).
*h)* $M \; sr_N \; M' :\Leftrightarrow \exists t \in T : M[\{t\} > M'$
($M'$ is sequentially reachable from $M$ in one step in $N$);
*i)* $M \; SR_N M' :\Leftrightarrow M \; (sr_N \cup {sr_N}^{-1})^*_{\mathcal{P}(\mathcal{S})} \; M'$
($M'$ is sequentially reachable from $M$ in $N$).

□

## Fundamental Situations

**Definition D53** Let $N = (S, T, F)$ be a net and $t, t' \in T$ and $M \subseteq S$.

a) *$t$ is enabled at $M$* $:\Leftrightarrow {}^\bullet t_F \subseteq M \wedge t^\bullet{}_F \cap M = \emptyset$;
b) *$t$ is reverse enabled at $M$* $:\Leftrightarrow t^\bullet{}_F \subseteq M \wedge {}^\bullet t_M \cap M = \emptyset$;
c) *$t$ has a contact at $M$* $:\Leftrightarrow {}^\bullet t_F \subseteq M \wedge t^\bullet{}_F \cap M \neq \emptyset$;
d) *$t$ has a reverse contact at $M$* $:\Leftrightarrow t^\bullet{}_F \subseteq M \wedge {}^\bullet t_F \cap M \neq \emptyset$;
e) *$t$ and $t'$ are in conflict at $M$* $:\Leftrightarrow$
$t$ and $t'$ are enabled at $M$ and $(t^\bullet{}_F \cup {}^\bullet t_F) \cap (t'^\bullet{}_F \cup {}^\bullet t'_F) \neq \emptyset$;
f) *$t$ and $t'$ are in reverse conflict at $M$* $:\Leftrightarrow$
$t$ and $t$' are reverse enabled at $M$ and $(t^\bullet{}_F \cup {}^\bullet t_F) \cap (t'^\bullet{}_F \cup {}^\bullet t'_F) \neq \emptyset$;
g) *$t$ has a transjunction at $M$* $:\Leftrightarrow {}^\bullet t_F \cap M \neq \emptyset \wedge t^\bullet{}_F \cap M \neq \emptyset$.

□

## Elementary Net Systems

**Definition D54** *$(S, T, F, C)$ is an elementary net systems (ENS)* $:\Leftrightarrow$
$(S, T, F)$ is a net and $C \subseteq S$. □

**Definition D55**
Let $NS = (S, T, F, C)$ be an elementary net system and $N = (S, T, F)$.

a) $CaseClass(NS) := [C]_{R_N}$ (*the case class of $NS$*);
b) $SeqCaseClass(NS) := [C]_{SR_N}$ (*the sequential case class of $NS$*).

□

**Definition D56** Let $NS = (S, T, F, C)$ be an elementary net system.

a) $ProperS(NS) := (\bigcup CaseClass(NS)) - (\bigcap CaseClass(NS))$
(*the proper $S$-elements of $NS$*);
b) $ProperT(NS) := \bigcup \{e : \exists m, m' \in CaseClass(NS) : m[e > m'\}$
(*the proper $T$-elements of $NS$*);
c) $NS$ is proper $:\Leftrightarrow S = ProperS(NS) \wedge T = ProperT(NS)$.

□

**Definition D57** Let $NS = (S, T, F, C_0)$ be an elementary net system.

a) $NS$ is safe $:\Leftrightarrow \forall M \in CaseClass(NS) : \neg\exists t \in T :$
   ($t$ has a contact or a reverse contact at $M$);
b) $NS$ is secure $:\Leftrightarrow$ $NS$ is safe and $\forall M \in CaseClass(NS) : \neg\exists t \in T :$
   ($t$ has a transjunction at $M$).

□

# B Models of Concurrency Theory

**The concurrency structure of fig. 1.a:**

$$
\begin{aligned}
X &= \{1,2,3,4,5,6,7,8,9,10,11,12\} \\
co &= \updownarrow \{(9,1),(9,2),(9,3),(10,3),(10,4),(10,5),(10,9),(11,5),(11,6),(11,7), \\
&\quad (11,10),(12,1),(12,7),(12,8),(12,9),(12,11)\} \\
li &= \updownarrow \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2), \\
&\quad (6,3),(6,4),(6,5),(7,1),(7,2),(7,3),(7,4),(7,5),(7,6),(8,1),(8,2),(8,3), \\
&\quad (8,4),(8,5),(8,6),(8,7),(9,4),(9,5),(9,6),(9,7),(9,8),(10,1),(10,2),(10,6), \\
&\quad (10,7),(10,8),(11,1),(11,2),(11,3),(11,4),(11,8),(11,9),(12,2),(12,3), \\
&\quad (12,4),(12,5),(12,6),(12,10)\} \\
\tilde{co}_X &= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9),(10,10),(11,11),(12,12)\} \\
\tilde{li}_X &= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9),(10,10),(11,11),(12,12)\} \\
Cuts(CS) &= \{\{1,9,12\},\{2,9\},\{3,9,10\},\{4,10\},\{5,10,11\},\{6,11\},\{7,11,12\},\{8,12\}\} \\
Lines(CS) &= \{\{1,2,3,4,5,6,7,8\},\{1,2,3,4,8,11\},\{1,2,6,7,8,10\},\{2,3,4,5,6,12\}, \\
&\quad \{2,6,10,12\},\{4,5,6,7,8,9\},\{4,8,9,11\}\} \\
P_{CS} &= \{(1,2),(1,8),(3,2),(3,4),(5,4),(5,6),(7,6),(7,8),(9,4),(9,8),(10,2),(10,6), \\
&\quad (11,4),(11,8),(12,2),(12,6)\} \\
D_{CS} &= \{(1,9),(1,12),(2,9),(3,9),(3,10),(4,10),(5,10),(5,11),(6,11),(7,11),(7,12), \\
&\quad (8,12)\} \\
Dom(P) &= \{1,3,5,7,9,10,11,12\} \\
Ran(P) &= \{2,4,6,8\} \\
Dom(D) &= \{1,2,3,4,5,6,7,8\} \\
Ran(D) &= \{9,10,11,12\}
\end{aligned}
$$

**The concurrency structure of fig. 2.a:**

$$
\begin{aligned}
X &= \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24\} \\
co &= \updownarrow \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3),(5,2),(5,4),(6,1),(6,3),(6,5), \\
&\quad (9,8),(10,1),(10,3),(10,5),(10,7),(10,8),(10,9),(11,8),(11,10), \\
&\quad (12,7),(12,9),(12,11),(13,8),(13,10),(13,12),(14,8),(14,10),(14,12), \\
&\quad (14,13),(15,7),(15,9),(15,11),(15,13),(15,14),(16,7),(16,9),(16,11), \\
&\quad (16,13),(16,14),(16,15),(17,14),(17,16),(18,13),(18,15),(18,17),(19,3), \\
&\quad (19,4),(19,14),(19,16),(19,18),(20,1),(20,2),(20,14),(20,16),(20,18), \\
&\quad (20,19),(21,3),(21,4),(21,13),(21,15),(21,17),(21,19),(21,20),(22,1), \\
&\quad (22,2),(22,13),(22,15),(22,17),(22,19),(22,20),(22,21),(23,3),(23,4), \\
&\quad (23,20),(23,22),(24,1),(24,2),(24,19),(24,21),(24,23)\} \\
li &= \updownarrow \{(5,1),(5,3),(6,2),(6,4),(7,1),(7,3),(7,5),(8,1),(8,3),(8,5),(9,2),(9,4), \\
&\quad (9,6),(10,2),(10,4),(10,6),(11,1),(11,2),(11,3),(11,4),(11,5),(11,6),
\end{aligned}
$$

$$
\begin{aligned}
& (11,7),(11,9),(12,1),(12,2),(12,3),(12,4),(12,5),(12,6),(12,8),(12,10),\\
& (13,1),(13,2),(13,3),(13,4),(13,5),(13,6),(13,7),(13,9),(13,11),(14,1),\\
& (14,2),(14,3),(14,4),(14,5),(14,6),(14,7),(14,9),(14,11),(15,1),(15,2),\\
& (15,3),(15,4),(15,5),(15,6),(15,8),(15,10),(15,12),(16,1),(16,2),(16,3),\\
& (16,4),(16,5),(16,6),(16,8),(16,10),(16,12),(17,1),(17,2),(17,3),(17,4),\\
& (17,5),(17,6),(17,7),(17,8),(17,9),(17,10),(17,11),(17,12),(17,13),(17,15),\\
& (18,1),(18,2),(18,3),(18,4),(18,5),(18,6),(18,7),(18,8),(18,9),(18,10),\\
& (18,11),(18,12),(18,14),(18,16),(19,1),(19,2),(19,5),(19,6),(19,7),(19,8),\\
& (19,9),(19,10),(19,11),(19,12),(19,13),(19,15),(19,17),(20,3),(20,4),(20,5),\\
& (20,6),(20,7),(20,8),(20,9),(20,10),(20,11),(20,12),(20,13),(20,15),(20,17),\\
& (21,1),(21,2),(21,5),(21,6),(21,7),(21,8),(21,9),(21,10),(21,11),(21,12),\\
& (21,14),(21,16),(21,18),(22,3),(22,4),(22,5),(22,6),(22,7),(22,8),(22,9),\\
& (22,10),(22,11),(22,12),(22,14),(22,16),(22,18),(23,1),(23,2),(23,5),(23,6),\\
& (23,7),(23,8),(23,9),(23,10),(23,11),(23,12),(23,13),(23,14),(23,15),(23,16),\\
& (23,17),(23,18),(23,19),(23,21),(24,3),(24,4),(24,5),(24,6),(24,7),(24,8),\\
& (24,9),(24,10),(24,11),(24,12),(24,13),(24,14),(24,15),(24,16),(24,17),\\
& (24,18),(24,20),(24,22)\}\\
\tilde{co}_X &= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9),(10,10),\\
& (11,11),(12,12),(13,13),(14,14),(15,15),(16,16),(17,17),(18,18),(19,19),\\
& (20,20),(21,21),(22,22),(23,23),(24,24)\}\\
\tilde{li}_X &= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9),(10,10),(11,11),\\
& (12,12),(13,13),(14,14),(15,15),(16,16),(17,17),(18,18),(19,19),(20,20),\\
& (21,21),(22,22),(23,23),(24,24)\}\\
Cuts(CS) &= \{\{1,2,3,4\},\{1,2,20,22\},\{1,2,24\},\{1,3,6\},\{1,3,9,10\},\{2,4,5\},\\
& \{2,4,7,8\},\{3,4,19,21\},\{3,4,23\},\{5,6\},\{5,9,10\},\{6,7,8\},\{7,8,9,10\},\\
& \{7,9,12\},\{7,9,15,16\},\{8,10,11\},\{8,10,13,14\},\{11,12\},\{11,15,16\},\\
& \{12,13,14\},\{13,14,15,16\},\{13,15,18\},\{13,15,21,22\},\{14,16,17\},\\
& \{14,16,19,20\},\{17,18\},\{17,21,22\},\{18,19,20\},\{19,20,21,22\},\\
& \{19,21,24\},\{20,22,23\},\{23,24\}\}\\
Lines(CS) &= \{\{1,5,7,11,13,17,19,23\},\{1,5,7,11,14,18,21,23\},\\
& \{1,5,8,12,15,17,19,23\},\{1,5,8,12,16,18,21,23\},\{2,6,9,11,13,17,19,23\},\\
& \{2,6,9,11,14,18,21,23\},\{2,6,10,12,15,17,19,23\},\{2,6,10,12,16,18,21,23\},\\
& \{3,5,7,11,13,17,20,24\},\{3,5,7,11,14,18,22,24\},\{3,5,8,12,15,17,20,24\},\\
& \{3,5,8,12,16,18,22,24\},\{4,6,9,11,13,17,20,24\},\{4,6,9,11,14,18,22,24\},\\
& \{4,6,10,12,15,17,20,24\},\{4,6,10,12,16,18,22,24\}\}\\
P_{CS} &= \{(1,5),(1,23),(2,6),(2,23),(3,5),(3,24),(4,6),(4,24),(7,5),(7,11),\\
& (8,5),(8,12),(9,6),(9,11),(10,6),(10,12),(13,11),(13,17),(14,11),(14,18),\\
& (15,12),(15,17),(16,12),(16,18),(19,17),(19,23),(20,17),(20,24),(21,18),
\end{aligned}
$$

$$
\begin{aligned}
& (21,23),(22,18),(22,24)\} \\
D_{CS} &= \emptyset \\
Dom(P) &= \{1,2,3,4,7,8,9,10,13,14,15,16,19,20,21,22\} \\
Ran(P) &= \{5,6,11,12,17,18,23,24\}
\end{aligned}
$$

**The concurrency structure of fig. 6:**

$$
\begin{aligned}
X &= \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\} \\
co &= \updownarrow \{(11,1),(11,2),(11,3),(12,3),(12,4),(12,5),(12,11),(13,5),(13,6), \\
&\quad (13,7),(13,12),(14,7),(14,8),(14,9),(14,13),(15,1),(15,9),(15,10), \\
&\quad (15,11),(15,14),(16,1),(16,7),(16,8),(16,9),(16,10),(16,11),(16,13), \\
&\quad (16,14),(16,15)\} \\
li &= \updownarrow \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2), \\
&\quad (6,3),(6,4),(6,5),(7,1),(7,2),(7,3),(7,4),(7,5),(7,6),(8,1),(8,2),(8,3), \\
&\quad (8,4),(8,5),(8,6),(8,7),(9,1),(9,2),(9,3),(9,4),(9,5),(9,6),(9,7),(9,8), \\
&\quad (10,1),(10,2),(10,3),(10,4),(10,5),(10,6),(10,7),(10,8),(10,9),(11,4), \\
&\quad (11,5),(11,6),(11,7),(11,8),(11,9),(11,10),(12,1),(12,2),(12,6),(12,7), \\
&\quad (12,8),(12,9),(12,10),(13,1),(13,2),(13,3),(13,4),(13,8),(13,9),(13,10), \\
&\quad (13,11),(14,1),(14,2),(14,3),(14,4),(14,5),(14,6),(14,10),(14,11),(14,12), \\
&\quad (15,2),(15,3),(15,4),(15,5),(15,6),(15,7),(15,8),(15,12),(15,13),(16,2), \\
&\quad (16,3),(16,4),(16,5),(16,6),(16,12)\} \\
Cuts(CS) &= \{\{1,11,15,16\},\{2,11\},\{3,11,12\},\{4,12\},\{5,12,13\},\{6,13\}, \\
&\quad \{7,13,14,16\},\{8,14,16\},\{9,14,15,16\},\{10,15,16\}\} \\
Lines(CS) &= \{\{1,2,3,4,5,6,7,8,9,10\},\{1,2,3,4,5,6,10,14\},\{1,2,3,4,8,9,10,13\}, \\
&\quad \{1,2,6,7,8,9,10,12\},\{1,2,6,10,12,14\},\{2,3,4,5,6,7,8,15\}, \\
&\quad \{2,3,4,5,6,16\},\{2,3,4,8,13,15\},\{2,6,7,8,12,15\},\{2,6,12,16\}, \\
&\quad \{4,5,6,7,8,9,10,11\},\{4,5,6,10,11,14\},\{4,8,9,10,11,13\}\} \\
P_{CS} &= \{(1,2),(1,10),(3,2),(3,4),(5,4),(5,6),(7,6),(7,8),(9,8),(9,10),(11,4), \\
&\quad (11,10),(12,2),(12,6),(13,4),(13,8),(14,6),(14,10),(15,2),(15,8),(16,2),(16,6)\} \\
D_{CS} &= \{(1,11),(1,15),(1,16),(2,11),(3,11),(3,12),(4,12),(5,12),(5,13),(6,13), \\
&\quad (7,13),(7,14),(7,16),(8,14),(8,16),(9,14),(9,15),(9,16),(10,15),(10,16), \\
&\quad (14,16),(15,16)\} \\
Dom(P) &= \{1,3,5,7,9,11,12,13,14,15,16\} \\
Ran(P) &= \{2,4,6,8,10\} \\
Dom(D) &= \{1,2,3,4,5,6,7,8,9,10,14,15\} \\
Ran(D) &= \{11,12,13,14,15,16\}
\end{aligned}
$$