Intersecting Multisets and Applications to Multiset Languages

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Abstract. We show that the family $\mathcal{S}em$ = **mREG** = **mCF** of regular multiset languages is closed under applications of finite and iterated multiset-union, finite and iterated multiset-intersection, and multiset-subtraction. For the class $\mathcal{S}em(\mathbb{Z}^k)$ of semilinear subsets of $\mathbb{Z}^k$, or that of $\mathbb{N}^k$, $\mathcal{S}em := \bigcup_{k\geq 0} \mathcal{S}em(\mathbb{N}^k)$, this amounts to verify, that the component-wise maximum, minimum or non-negative subtraction of pairs of elements from two semi-linear sets is again semi-linear, and that the iterated application can be replaced by a fixed finite application of multiset-intersection, respectively multiset-subtraction.

We solve the three questions about closure properties that remained open in [KuMi 01,KuMi 02], verify that the family **mMON** is not closed with respect to multiset-intersection, and correct a small mistake in a proof in [EiSc 69,Bers 79].

1 Introduction

The interest in multisets and subsets of commutative monoids has increased in the last years. This is described for instance in [HePP 97], [KuPV 01,KuMi 01,KuMi 02], and many others not cited here. In standard formal language theory, see [Gins 75,DaPă 89], already some results have been obtained for commutative strings and languages thereof in [CrMa 76,Latt 79,Kort 80], to name a few. In comparison, the arbitrary multiset grammars of [KuPV 01,KuMi 01,KuMi 02] are in some sense equivalent to variants of vector replacement systems, or Petri nets, see e.g. [JaVa 80,Pete 81,Jaff 77,Card 75]. However, the use of Petri nets is of quite different nature.

In the context of multiset grammars, one began to collect results that parallel those of standard formal language theory or help to clarify the difference. For example, the operation of multiset intersection has no adequate counterpart for strings in a non-commutative monoid. Rational subsets of commutative monoids, on the other hand, have been studied for a long time in [Pres 30,EiSc 69,GiSp 64,GiSp 65,Jaff 77], but the question of taking the componentwise minimum, maximum, or positive subtraction was not attacked in those papers, and these problems arose when considering finite multisets, that are equivalent to vector sets. Using the results from [Biry 67,EiSc 69], see also [Card 75,Jaff 77], we can solve the open questions from [KuPV 01,KuMi 01,KuMi 02] without difficulty.

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2 Basic Definitions

Multisets over a domain D are, in all generality, total mappings $m : D \to \mathbb{N} \cup \{\infty\}$, sometimes written as $m \in (\mathbb{N} \cup \{\infty\})^D$, where $\mathbb{N}$ denotes the non-negative integers. The value $m(d)$ gives the number of copies of the element d in m, and $m(d) = \infty$ means, that d occurs infinitely often within m. The usual addition is extended to $\mathbb{N} \cup \{\infty\}$ by the obvious $\infty + x := x + \infty := \infty$ for each $x \in \mathbb{N} \cup \{\infty\}$. The *weight* of m is given by $|m| := \sum_{d \in D} m(d)$, and a multiset $m \in \mathbb{N}^D$ is finite, iff $|m| \in \mathbb{N}$.

A usual set can easily be represented as multiset $m \in \{0, 1\}^D$, that is, the mapping m is the characteristic function of the set, which is represented. The set of all subsets of the set M will be denoted by 2^M.

We will solely use finite multisets over a linearly ordered domain. For the domain $D := \{a_1, \ldots, a_n\}$, a multiset m can be represented by the vector $m \in \mathbb{N}^n$, for which $m(i)$ equals the number of occurrences of the element a_i within m. In order to avoid the zero entries in multisets with only a small number of different types of elements, we use the notion from [KuMi 02] and use equivalence classes of strings from D^* modulo the well known Parikh mapping. The Parikh mapping $\psi : D^* \to \mathbb{N}^{|D|}$ is a homomorphism and defined by $\psi(w)(d) := |w|_d$ for each $d \in D$, where $|w|_d$ denotes the number of occurrences of the symbol d within w. The class $[w] := \psi^{-1}(\psi(w)) = \{v \in D^* \mid \psi(v) = \psi(w)\}$ will represent the multiset m, where $m(d) = |w|_d$. Consequently, the empty multiset is denoted by $[\lambda]$.

Definition 1. *a) $D^{\circledast}$ denotes the set of all finite multisets over the domain D. Any set of multisets is called multiset language, thus $D^{\circledast}$ and all its subsets are multiset languages.*

b) Multiset-addition is defined for $m_1, m_2 \in D^{\circledast}$ by $(m_1 + m_2)(d) := m_1(d) + m_2(d)$ for each $d \in D$. For multiset languages $A, B \subseteq D^{\circledast}$ let $A + B := \{m_1 + m_2 \mid m_1 \in A, m_2 \in B\}$.

c) For any multiset language $A \subseteq D^{\circledast}$ we define its addition-closures $A^{\circledast} := A^{\oplus} \cup \{[\lambda]\}$, where $A^{\oplus} := \bigcup_{i \geq 1} A_i$, $A_{i+1} := A_i + A$ and $A_1 := A$.

d) Multiset-subtraction is defined by $\forall d \in D : (m_1 - m_2)(d) := max(0, m_1(d) - m_2(d))$.

e) Multiset-inclusion is defined as follows: $m_1 \sqsubseteq m_2$ iff $\forall d \in D : m_1(d) \leq m_2(d)$.

f.1) Multiset-union is defined by $\forall d \in D : (m_1 \sqcup m_2)(d) := max(m_1(d), m_2(d))$.

f.2) For multiset languages $A, B \subseteq D^{\circledast}$ let $A \vee B := \{(m_1 \sqcup m_2 \mid (m_1 \in A, m_2 \in B\}$.

f.3) For families $\mathcal{F}_1, \mathcal{F}_2$ of (multiset) languages let $\mathcal{F}_1 \vee \mathcal{F}_2 := \{(L_1 \cup L_2 \mid (L_1 \in \mathcal{F}_1, L_2 \in \mathcal{F}_2\}$.

g.1) Multiset-intersection is defined by $\forall d \in D : (m_1 \sqcap m_2)(d) := min(m_1(d), m_2(d))$.

g,2) For multiset languages $A, B \subseteq D^{\oplus}$ let $A \wedge B := \{(m_1 \sqcap m_2 \mid (m_1 \in A, m_2 \in B\}$.

g.3) For families $\mathcal{F}_1, \mathcal{F}_2$ of (multiset) languages let $\mathcal{F}_1 \wedge \mathcal{F}_2 := \{(L_1 \cap L_2 \mid (L_1 \in \mathcal{F}_1, L_2 \in \mathcal{F}_2\}$.

Remarks on Definition 2.1:

In many references, e.g. [GrSc 93,HePP 97], the term multiset-union is used instead of multiset-addition, but our definition of multiset-union coincides with set-union for those multisets that represent sets. If we identify each element $d \in D$ with the multiset $[d]$, then D is a multiset language, and the set $D^{\circledast}$ of all multisets over D is indeed the full addition-closure of this multiset language. In [KuMi 01,KuMi 02] no distinction has been made between the full addition-closure $A^{\circledast}$ and the (positive) addition-closure $A^{\oplus}$.

Since multiset languages (and families of multiset languages, as well as families of string languages) are sets, we have to use set-union and set-intersection also in their original meanings. Hence, as is well known in standard formal language theory, the *vee*($\vee$) and *wedge*($\wedge$) are used for element-wise union, respectively intersection, of the members in the multiset languages or families. Those can then be either multisets with the appropriate maximum or minimum interpretation or they are languages (of strings or multisets), where usual set-union resp. set-intersection has to be applied.

The set $D^{\circledast}$ of all multisets over D is a commutative monoid with $[\lambda]$ as neutral element and multiset-addition as operation. It is well accepted, that the rational subsets in any commutative monoid $(M, +, 0)$ are precisely the semi-linear subsets of M, see [EiSc 69,Bers 79], where, unfortunately, the proofs are a bit faulty[1]. We will correct this in what follows.

Definition 2. *Let $(M, +, 0)$ be any commutative monoid with two sided unit 0 and commutative, associative addition. The family $\mathcal{R}at(M)$ is the least family of subsets of M satisfying the following:*

- *(R1)* $\emptyset \in \mathcal{R}at(M)$,
- *(R2)* $\forall m \in M : \{m\} \in \mathcal{R}at(M)$,
- *(R3) If $A, B \in \mathcal{R}at(M)$, then also $A \cup B \in \mathcal{R}at(M)$,*
- *(R4) If $A, B \in \mathcal{R}at(M)$, then also $A + B \in \mathcal{R}at(M)$,*
- *(R5) If $A \in \mathcal{R}at(M)$, then also $A^{\oplus} \in \mathcal{R}at(M)$.*

Recall, that $A^{\circledast} := A^{\oplus} \cup \{0\}$ is the (commutative) submonoid generated by A, where $A^{\oplus}$ is defined as in c) of Def. 2.1, and addition is generalized for sets A and B as in b) of Def. 2.1. Since in a commutative monoid many equations can be simplified, for instance $(A \cup B)^{\circledast} = A^{\circledast} + B^{\circledast}$, the notion of semi-linear sets eases the discussion. The identification of rational subsets of M with the semi-linear subsets of M has been proven for free commutative monoids (M,+,0) in [GiSp 64,GiSp 65], but is valid also for finitely generated commutative monoids, that are not freely generated. From the results of Rédei, [Rede 63], we know, that each finitely generated commutative monoid can be finitely presented. The commutative monoid $(\mathbb{Z}^k, +, \mathbf{0})$ is such an example, that we will make use of. The monoid $(\mathbb{N}^k, +, \mathbf{0})$ is the free, commutative monoid on n generators, and isomorphic to the monoid $(D^{\circledast}, +, [\lambda])$ for any finite set $D := \{a_1, \ldots, a_n\}$. Thereby, each element a_i is in one-to-one correspondence with the i-th unit vector ${}^{\top}(0, \ldots 0, 1, 0, \ldots 0)$,

[1] In both proofs it was claimed (in their respective notation) that $(c + B^{\circledast})^{\circledast} = (\{c\} \cup B)^{\circledast}$. This is not correct, since in the former set no element from $B^{\circledast}$ appears without the constant c being added at least once.

denoted here in its transposed version as n-tuple, having the figure '1' at the i-th position. It is known, [EiSc 69,Bers 79], that $\mathcal{R}at(M)$ is a boolean algebra for any finitely generated monoid M. We will use this especially for $\mathcal{R}at(\mathbb{N}^k)$ and $\mathcal{R}at(\mathbb{Z}^k)$.

There are some more elementary properties of rational sets, see [EiSc 69,Bers 79], that shall be listed here for later use.

Theorem 1. *Let $(M, \cdot, 1)$, and $(M', \odot, e)$ be finitely generated commutative monoids.*

1. *For any homomorphism $h : M \to M'$, the set $h(R) \subseteq M'$ is rational for each rational subset R of M.*
2. *For any homomorphism $h : M \to M'$, the set $h^{-1}(R) \subseteq M'$ is rational for each rational subset R of M'.*
3. *If R_1 and R_2 are rational subsets of M, then also $R_1 \cap R_2$, $R_1 \times R_2$, and $R_1 \setminus R_2$ are rational subsets of M.*

In Theorem 2.1 the first entry is easily proved by applying the homomorphism to the elements within the rational expression for R, thus yielding the rational expression for R'. Moreover, 1. and 2. also hold for free and finitely generated, non-commutative monoids. 3. holds for all finitely generated monoids, which is Corollary III.1 in [EiSc 69], and was shown for finitely generated free commutative monoids (i.e. for $\mathbb{N}^k$) in [GiSp 64].

Definition 3. *Let $(M, +, 0)$ be a commutative monoid, $c \in M$, and $A \subseteq M$ finite, then $\{c\} + A^{\circledast}$ is called linear, and each finite union of linear sets is called semi-linear. We will omit braces whenever possible, and write $c + A^{\circledast}$ instead of $\{c\} + A^{\circledast}$.*

The family of semilinear subsets of a commutative monoid $(M, +, 0)$ is denoted by $\mathcal{S}em(M)$, and $\mathcal{S}em$ is used to denote the family of all semilinear subsets of $\mathbb{N}^k$ for all $k \in \mathbb{N}$.

Theorem 2. *$\mathcal{R}at(M) = \mathcal{S}em(M)$ holds for any commutative monoid $(M, +, 0)$.*

Proof: Obviously $\mathcal{S}em(M) \subseteq \mathcal{R}at(M)$. The converse is proved by structural induction: First, finite sets are semi-linear and the union of semi-linear sets is semi-linear again by definition. It remains to show, that semi-linear sets are closed under the elementwise sum and addition-closure. Since + distributes over finite unions, we only have to verify that the sum of two linear sets is semi-linear. In fact, we again obtain a linear set by the summation: $(c + A^{\circledast}) + (d + B^{\circledast}) = c + d + A^{\circledast} + B^{\circledast} = (c + d) + (A \cup B)^{\circledast}$. Now, to prove that addition closure of semi-linear sets yields always semi-linear sets, it is sufficient to show, that $(c + A^{\circledast})^{\circledast}$ is semi-linear, since $(C \cup D)^{\circledast} = C^{\circledast} + D^{\circledast}$. One verifies $(c + A^{\circledast})^{\circledast} = \{c\} + (\{c\} \cup B)^{\circledast} \cup \{0\}$. Together with the trivial equation $(A^{\circledast})^{\circledast} = A^{\circledast}$ the proof of Theorem 2.2 is completed.

Corollary 1. *For each $k \in \mathbb{N}$ we have $A \in \mathcal{S}em(\mathbb{Z}^k)$ and $B \in \mathcal{S}em(\mathbb{N}^k)$ implies $A \cap B \in \mathcal{S}em(\mathbb{N}^k)$.*

Proof: This follows from the first and third entry in Theorem 2.1 by using $(M, \cdot, 1) := (\mathbb{Z}^k, +, 0)$ and $(M', \cdot, 1) := (\mathbb{N}^k, +, 0)$: For $A \in \mathcal{R}at(\mathbb{Z}^k) = \mathcal{S}em(\mathbb{Z}^k)$ we find $C := A \cap \mathbb{N}^k \in \mathcal{R}at(\mathbb{Z}^k)$, since $\mathbb{N}^k$ is an element of the boolean algebra $\mathcal{R}at(\mathbb{Z}^k) = \mathcal{S}em(\mathbb{Z}^k)$. Hence, by Theorem 2.1, 1., $C \subseteq \mathbb{N}^k$ is $h(C)$ for the embedding homomorphism $h : \mathbb{Z}^k \to \mathbb{N}^k$, and thus a rational subset of $\mathbb{N}^k$, too. Since $\mathcal{R}at(\mathbb{N}^k) = \mathcal{S}em(\mathbb{N}^k)$ is closed with respect to intersection, we finally conclude $A \cap B = C \cap B \in \mathcal{S}em(\mathbb{N}^k)$.

3 New closure properties of multiset languages

Definition 4. *Let $\mathcal{R}eg$ (resp. $\mathcal{C}f$, $\mathcal{C}s$) denote the families of regular sets (context-free, context sensitive languages, respectively).*

It was verified in [KuMi 01,KuMi 02] that $\mathcal{S}em$ = **mREG**, where **mREG** denotes the family of multiset languages generated by regular multiset grammars. This followed from the well known Theorem of Parikh[2], and can also be deduced with the help of Theorem 2.2, from [EiSc 69]: If one studies the Parikh image of regular languages in $\mathcal{R}eg(\Sigma^*) = \mathcal{R}at(\Sigma^*)$, then one may modify any rational expression for a set $L \in \mathcal{R}at(\Sigma^*)$ by replacing the non-commutative product by the commutative addition operation, and the Kleene closure $()^*$ by the addition-closure $()^{\circledast}$. Working now in the commutative monoid $(\Sigma^{\oplus}, +, [\lambda])$ of multisets, the Parikh image for the language represented by the modified rational expression is identical with that of the former.

In [KuMi 01,KuMi 02] it was left open, whether the family $\mathcal{S}em(\mathbb{N}^k)$ is closed with respect to elementwise multiset-union or multiset-intersection, that is, whether $A \circledcirc B \in \mathcal{S}em(\mathbb{N}^k)$ for $A, B \in \mathcal{S}em(\mathbb{N}^k)$ and $\circledcirc \in \{\sqcup, \sqcap\}$.

Also the question, whether **mMON** or **PsCS** $= \psi(\mathcal{C}s)$ is closed under $\sqcap$, i.e., elementwise multiset-intersection, was not answered in [KuMi 01,KuMi 02].

The latter question can be answered easily:

Theorem 3. *The family* **mMON** *is not closed with respect to multiset-intersection, $\sqcap$.*

Proof: Let $A := \{[a], [b]\}$, then $B := A \sqcap A = \{[\lambda], [a], [b]\}$, but the empty multiset $[\lambda]$ cannot be generated from a non-empty axiom using monotone rewriting rules.

However, this proof is not really satisfying, since $A, B \in$ **mMON** might imply, that $(A \sqcap B) \setminus F \in$ **mMON** for some finite set F. As of now, we were not able to find a counter example for this statement.

In the following we shall proof that the family $\mathcal{S}em$ is closed w.r.t. multiset-union, multiset-intersection, and multiset-subtraction.

Theorem 4. *The families $\mathcal{R}at(\mathbb{Z}^k)$, and $\mathcal{R}at(\mathbb{N}^k)$ are closed with respect to $\sqcup$, $\sqcap$, and $-$ that is, for $A, B \in \mathcal{R}at(\mathbb{Z}^k)$ (or $A, B \in \mathcal{R}at(\mathbb{N}^k)$) we have $A \sqcup B \in \mathcal{R}at(\mathbb{Z}^k)$, $A \sqcap B \in \mathcal{R}at(\mathbb{Z}^k)$, and $A - B \in \mathcal{R}at(\mathbb{Z}^k)$ ($\in \mathcal{R}at(\mathbb{N}^k)$, respectively).*

For the proof, we will describe the sets $A \sqcup B$, $A \sqcap B$, and $A - B$ by applying operations to sets $A, B \in \mathcal{R}at(\mathbb{Z}^k)$ (respectively $A, B \in \mathcal{R}at(\mathbb{N}^k)$), with respect to which the family $\mathcal{R}at$ ($\mathcal{S}em$, resp.) is closed.

Proof: Let $A, B \in \mathcal{R}at(\mathbb{Z}^k)$ (respectively $A, B \in \mathcal{R}at(\mathbb{N}^k)$), then by Theorem 2. 1, 3., $A \times B \in \mathcal{R}at(\mathbb{Z}^{2k})$ ($A \times B \in \mathcal{R}at(\mathbb{N}^{2k})$). For an easier reading, we will write the elements of $A \times B$ as matrix with two columns, i.e., $A \times B \in \mathcal{R}at(\mathbb{Z}^{k \times 2})$. Now we use the linear mapping $\varphi : \mathbb{Z}^{k \times 2} \to \mathbb{Z}^{k \times 4}$ to obtain the vectors ${}^{\top}(a_1 - b_1, a_2 - b_2, \ldots a_k - b_k)$ and ${}^{\top}(b_1 - a_1, b_2 - a_2, \ldots b_k - a_k)$ from which we retrieve the information about minimum or maximum: $max(a_i, b_i) = a_i$, iff $a_i - b_i \in \mathbb{N}$. Since $A \times B$ is rational and φ is a homomorphism, it follows that $\varphi(A \times B)$ is rational, too. $\varphi(A \times B)$ is defined

[2] $\psi(L) \in \mathcal{S}em$ for each context-free (or regular) set $L \in \mathcal{C}f$ (resp. $L \in \mathcal{R}eg$)

by multiplying each element of $A \times B$, represented by a $(k \times 2)$-matrix from $\mathbb{Z}^{k\times 2}$ (or $\mathbb{N}^{k\times 2}$) with the (2×4)-matrix $\varphi := \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$. For $\mathbf{X} := \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_k & b_k \end{pmatrix}$ this gives $\mathbf{Y} := \varphi(\mathbf{X}) \in \mathbb{Z}^{k\times 4}$ by:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_k & b_k \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & a_1 - b_1 & b_1 - a_1 \\ a_2 & b_2 & a_2 - b_2 & b_2 - a_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_k & b_k & a_k - b_k & b_k - a_k \end{pmatrix}.$$

The resulting rational set $C := \varphi(A \times B) = \{\mathbf{Y} \mid \mathbf{Y} = \mathbf{X} \cdot \varphi \text{ for } \mathbf{X} \in A \times B\} \in \mathcal{R}at(\mathbb{Z}^{k\times 4})$ will then be intersected with an appropriate rational set T_r, $1 \leq r \leq 2^k$, that selects non-negative entries of the last two columns, and is to be followed by an projection $\pi_{r,max} : \mathbb{Z}^{k\times 4} \to \mathbb{Z}^k$ (or similar $\pi_{r,min}$, $\pi_{r,subtr}$). There exist 2^k different possibilities to allow a non-negative entry in one of the two last columns of an element (matrix) $\mathbf{Y} \in C$: The i-th row of $\mathbf{Y}$ is either an element of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \times \mathbb{Z}$, which means that $max(a_i, b_i) = a_i$, or of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{N}$, which means that $max(a_i, b_i) = b_i$. For each of the 2^k possible selections we define the rational set $T_r \subseteq \mathbb{Z}^{k\times 4}$ and the projections $\pi_{r,max}$, $\pi_{r,min}$, and $\pi_{r,subtr}$ that follow the intersection $D_r := C \cap T_r$. These projections are defined separately for each of the rows of T_r as follows:

If the i-th row of T_r is $\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}\times\mathbb{Z}$, then $\pi_{r,max}$ is the projection onto the first column of this row of D_r, yielding $\pi_{r,max}(D_r)(i) = D_r(i, 1) = a_i$. Likewise, $\pi_{r,min}$ is the projection onto the second column of this row of D_r, yielding $\pi_{r,min}(D_r)(i) = D_r(i, 2) = b_i$, and $\pi_{r,subtr}$ is the projection onto the third column of this row of D_r, yielding $\pi_{r,subtr}(D_r)(i) = D_r(i, 3) = a_i - b_i$.

And if the i-th row of T_r is $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}$, then $\pi_{r,max}$ is the projection onto the second column of this row of D_r, yielding $\pi_{r,max}(D_r)(i) = D_r(i, 2) = b_i$. Likewise, $\pi_{r,min}$ is the projection onto the first column of this row of D_r, yielding $\pi_{r,min}(D_r)(i) = D_r(i, 1) = a_i$, and $\pi_{r,subtr}$ is the fix-projection onto zero, yielding $\pi_{r,subtr}(D_r)(i) = 0$.

It follows, that $A \sqcap B = \bigcup\limits_{1\leq r\leq 2^k} \pi_{r,max}(D_r)$ is a rational subset of $\mathbb{Z}^k$, as well as are the sets $A \sqcup B = \bigcup\limits_{1\leq r\leq 2^k} \pi_{r,min}(D_r)$, and $A - B = \bigcup\limits_{1\leq r\leq 2^k} \pi_{r,subtr}(D_r)$.

This proof is valid also for the case of ordinary semilinear sets $A, B \in \mathcal{R}at(\mathbb{N}^k)$. For this conclusion, we start with semi-linear sets $A, B \in \mathcal{R}at(\mathbb{N}^k) \subseteq \mathcal{S}em$, which are considered as rational subsets of $\mathbb{Z}^k$. We then do the transformations in the proof of Theorem 2.4, yielding $A \sqcup B \in \mathcal{R}at(\mathbb{Z}^k)$, but $A \sqcup B \subseteq \mathbb{N}^k$ implies $A \sqcup B \in \mathcal{R}at(\mathbb{N}^k)$ by Corollary 2.1. The same argumentation can be used for $A \sqcap B$ and $A - B$.

Having shown, that semi-linear subsets of $\mathbb{Z}^k$ and of $\mathbb{N}^k$ are closed under $\sqcup, \sqcap$, and positive subtraction, we can in addition conclude, that also the indefinite iteration $A^{\sqcup}$, or $A^{\sqcap}$ of a semi-linear set $A \subseteq \mathbb{N}^k$ remains semi-linear. These closure operations are defined below by the obvious method.

Definition 5. *The iterated multiset-union $A^{\sqcup}$,and iterated multiset-intersection $A^{\sqcap}$ of a set $A \in \mathbb{N}^k$ is defined by:*

a) $A^{\sqcup} := \bigcup_{i\geq 1} A_i$*, where* $A_1 := A$*, and* $A^{(i+1)\sqcup} := A^{(i)\sqcup} \sqcup A$.

b) $A^{\sqcap} := \bigcup_{i\geq 1} A_i$*, where* $A_1 := A$*, and* $A^{(i+1)\sqcap} := A^{(i)\sqcap} \sqcap A$.

Lemma 1. *The family* $\mathcal{S}em$ *of semi-linear sets is closed with respect to iterated multiset union and iterated multiset intersection. That is, the sets* $A^{\sqcup}$*, and* $A^{\sqcap}$ *are semi-linear for each* $A \in \mathcal{S}em$*.*

Proof: Let $A \in \mathcal{R}at(\mathbb{N}^k) \subseteq \mathcal{S}em$, then we show $A^{\sqcup} = A^{(k)\sqcup}$, from which the result follows from Theorem 2.4. For iterated multiset intersection we replace $\sqcup$ by $\sqcap$, everywhere.

First we observe $A \subseteq A \sqcup A$, from which $A^{(i)\sqcup} \subseteq A^{(i+1)\sqcup}$ follows for each $i \geq 1$. Now, let $\mathbf{a} := (a_1, a_2, \ldots, a_k) \in A^{\sqcup}$ be arbitrary, then $\mathbf{a} = \mathbf{m}_1 \sqcup \mathbf{m}_2 \sqcup \ldots \sqcup \mathbf{m}_{k-1} \sqcup \mathbf{m}_k$, for k vectors $\mathbf{m}_j$, $(1 \leq i \leq k)$ having the maximum a_i as their i-th component. Hence, $A^{\sqcup} \subseteq A^{(k)\sqcup}$ and the result is proven for iterated multiset union. For iterated multiset intersection recall the above remark.

From Theorem 2.4, we conclude as corollary the solution of the remaining open question in the table to Theorem 6.2 in [KuMi 01], or in that to Theorem 3.3 in [KuMi 02].

Corollary 2. *The family* $\mathcal{S}em$ = **mREG** = **mCF** *is closed under applications of multiset-union, multiset-intersection, and multiset-subtraction.*

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